The Small Open Economy in a Generalized Gravity Model

Online Appendix

Svetlana Demidova
McMaster U

Konstantin Kucheryavyy
CUNY Baruch College

Takumi Naito
Waseda U

Andrés Rodríguez-Clare
UC Berkeley and NBER

August 2024

1 Microfoundations of the Multi-Sector Gravity Model

In this appendix, we present five setups, four of which can be used as microfoundations for the multi-sector gravity model introduced in Section 5 of the main text. These setups include the Armington and Eaton-Kortum models with external economies of scale, a generalized Krugman model, and a generalized Melitz-Pareto model with fixed costs paid in labor of source countries. The fifth setup, a generalized Melitz-Pareto model with fixed costs paid in labor of destination countries, does not align with the structure of the multi-sector gravity model discussed in Section 5. However, alongside the other four models, it serves as a suitable microfoundation for the single-sector gravity model detailed in Section 2. Hence, we introduce a generalized Melitz-Pareto-destination model in this appendix, explain why it does not fit the framework of the multi-sector gravity model, and subsequently demonstrate in Appendix 2.1 how it functions as a microfoundation for the single-sector gravity model. In Appendix 2 we also show how to obtain the single-sector versions of the other four setups introduced in Section 2.

Each of the five setups is mostly standard, and so we omit many details. For brevity, we often refer to sector k of country i as sector (i, k). Also, we use the terms "sector" and "industry" interchangeably.

Each of the five setups has a different specification for the sector-level consumption indices. We denote the consumption index in sector (j,k) as $C_{j,k}$. Above the level of these indices, the demand structure is the same across all five microfoundations. Specifically, the sector-level consumption indices are aggregated using a Cobb–Douglas utility function $U_j = \prod_{k=1}^{K} C_{j,k}^{\beta_{j,k}}$ with $\sum_{k=1}^{K} \beta_{j,k} = 1$. Consequently, the sectoral expenditure $X_{j,k} \equiv P_{j,k} C_{j,k}$ is a constant fraction $\beta_{j,k}$ of the total expenditure $X_j \equiv P_j U_j$, where $P_{j,k}$ and $P_j \equiv \prod_{k=1}^{K} (P_{j,k}/\beta_{j,k})^{\beta_{j,k}}$ are the price indices associated with $C_{j,k}$ and U_j, respectively. In all five microfoundations, we denote by $X_{ij,k}$ the total expenditure by consumers in j on goods from sector (i,k). The corresponding expenditure (or trade) share is given by $\lambda_{ij,k} = X_{ij,k}/X_{j,k}$.

All five setups feature three policy instruments: import tariffs, export taxes, and employment subsidies. It helps to describe these policy instruments using consistent terminology across the setups. Describing the employment subsidy is straightforward: each unit of labor employed in industry (i,k) receives a subsidy of $s_{i,k} < 1$ from country-i's government. Thus, the total employment subsidy in industry (i,k) equals $(1 - \bar{s}_{i,k})\, w_i L_{i,k}$, where $\bar{s}_{i,k} \equiv 1 - s_{i,k}$, $L_{i,k}$ is the total amount of labor employed in industry (i,k), and w_i is the wage in country i.

To explain import tariffs and export taxes, let $p_{ij,k}$ be the price paid by consumers in country j for a good produced in industry (i,k). Before the good from industy (i,k) reaches consumers in j, it traverses two borders and is consequently subject to two taxes: country j's import tariff, $t^m_{ij,k} > -1$, and country i's export tax $t^x_{ij,k} < 1$.

To elucidate, working backwards from country j to country i, before the good from industry (i,k) crosses the border with country j, its price equals $p_{ij,k}/\bar{t}^m_{ij,k}$, where $\bar{t}^m_{ij,k} \equiv 1 + t^m_{ij,k}$. The government of country j collects the import tariff of $t^m_{ij,k} p_{ij,k}/\bar{t}^m_{ij,k}$ from this price. At the preceding step, before the good from industry (i,k) crosses the border of country i, its price equals $\bar{t}^x_{ij,k} p_{ij,k}/\bar{t}^m_{ij,k}$, with $\bar{t}^x_{ij,k} \equiv 1 - t^x_{ij,k}$. The government of country i collects the export tax of $t^x_{ij,k} p_{ij,k}/\bar{t}^m_{ij,k}$ from this price.

Given the above policy instruments, we can describe tax revenues of each country j. On the import side, country j's expenditure on goods from industry (i,k) evaluated at pre-tariff import prices is given by $X_{ij,k}/\bar{t}^m_{ij,k}$. This spending generates revenue for country j through import tariffs, which is given by $\left(\bar{t}^m_{ij,k} - 1\right) X_{ij,k}/\bar{t}^m_{ij,k}$. On the export side, country i purchases goods from industry (j,k), and the associated country j's revenues from export taxes are given by $\left(1 - \bar{t}^x_{ji,k}\right) X_{ji,k}/\bar{t}^m_{ji,k}$. Therefore, given that the employment subsidies in country j are given by $(1 - \bar{s}_{j,k})\, w_j L_{j,k}$, the aggregate tax revenues for country

j are calculated as

$$T_j = \sum_k \left\{ \sum_i \left[\frac{\bar{t}^m_{ij,k} - 1}{\bar{t}^m_{ij,k}} X_{ij,k} + \frac{1 - \bar{t}^x_{ji,k}}{\bar{t}^m_{ji,k}} X_{ji,k} \right] - \left(1 - \bar{s}_{j,k}\right) w_j L_{j,k} \right\}.$$

Finally, in all five setups, the total expenditure X_j of consumers in country j is equal to their income, which in turn consists of the wage income and the tax revenue of the government, $X_j = w_j L_j + T_j$.

1.1 An Armington Model with External Economies of Scale

Each country i produces a unique good in each sector k and sells this good to all other countries. The consumption index $C_{j,k}$ in industry (j,k) is obtained by combining goods produced by all countries i using a Constant Elasticity of Substitution aggregator with the elasticity of substitution $\eta_k > 1$.

All markets are perfectly competitive. Production technology features external economies of scale. Formally, firms from industry (i,k) perceive labor productivity as exogenously given by $\overline{A}_{i,k} L^{\gamma_k}_{i,k}$, where $\overline{A}_{i,k}$ is the exogenous component of productivity, $L_{i,k}$ is the total amount of labor employed in industry (i,k), and γ_k is the scale elasticity.

Let $P_{ij,k}$ be the price paid by consumers in country j for the good produced in industry (i,k). Given the import and export taxes, a firm from industry (i,k) selling to j chooses its output $y_{ij,k}$ so as to maximize its profits given by

$$\pi_{ij,k} = \frac{\bar{t}^x_{ij,k} P_{ij,k}}{\bar{t}^m_{ij,k}} y_{ij,k} - \bar{s}_{i,k} w_i \frac{\overline{\tau}_{ij,k} y_{ij,k}}{\overline{A}_{i,k} L^{\gamma_k}_{i,k}},$$

where $\bar{s}_{i,k} \equiv 1 - s_{i,k}$ is one minus i's employment subsidy in sector k, and $\overline{\tau}_{ij,k}$ is the iceberg trade cost. Therefore, under perfect competition, price $P_{ij,k}$ is given by

$$P_{ij,k} = \frac{\bar{t}_{ij,k} \overline{\tau}_{ij,k} w_i}{\overline{A}_{i,k} L^{\gamma_k}_{i,k}}, \tag{1}$$

where $\bar{t}_{ij,k} \equiv \bar{t}^m_{ij,k} \bar{s}_{i,k} / \bar{t}^x_{ij,k}$ is the aggregate policy index of trade from i to j in sector k.

Country j's expenditure share on goods from industry (i,k) is given by

$$\lambda_{ij,k} = \left(\frac{P_{ij,k}}{P_{j,k}} \right)^{1-\eta_k} = \frac{\left(\bar{t}_{ij,k} \overline{\tau}_{ij,k} w_i / \overline{A}_{i,k} \right)^{-(\eta_k - 1)} L^{(\eta_k - 1)\gamma_k}_{i,k}}{\sum_l \left(\bar{t}_{lj,k} \overline{\tau}_{lj,k} w_l / \overline{A}_{l,k} \right)^{-(\eta_k - 1)} L^{(\eta_k - 1)\gamma_k}_{l,k}},$$

where

$$P_{j,k} = \left[\sum_i P_{ij,k}^{1-\eta_k}\right]^{\frac{1}{1-\eta_k}} = \left[\sum_i \left(\bar{t}_{ij,k}\bar{\tau}_{ij,k}w_i/\overline{A}_{i,k}\right)^{-(\eta_k-1)} L_{i,k}^{(\eta_k-1)\gamma_k}\right]^{-\frac{1}{\eta_k-1}}. \tag{2}$$

The goods market clearing condition for industry (i,k) is given by

$$\bar{s}_{i,k}w_iL_{i,k} = \sum_j \frac{\bar{t}_{ij,k}^x}{\bar{t}_{ij,k}^m}\lambda_{ij,k}\beta_{j,k}\left(w_jL_j + T_j\right),$$

which can also be written as

$$w_iL_{i,k} = \sum_j \left(\lambda_{ij,k}/\bar{t}_{ij,k}\right)\beta_{j,k}\left(w_jL_j + T_j\right).$$

Finally, the labor market clearing condition in country i is given by $\sum_k L_{i,k} = L_i$.

1.2 An Eaton-Kortum Model with External Economies of Scale

Each industry k consists of a continuum of varieties $\omega \in \Omega_k \equiv [0,1]$. Any country has a technology to produce each of these varieties, and versions of any variety ω produced by different countries are perfect substitutes. The consumption index $C_{j,k}$ in industry (j,k) is obtained by combining consumption of all varieties ω using a Constant Elasticity of Substitution aggregator with the elasticity of substitution $\sigma_k > 1$.

All markets are perfectly competitive. Production technology features external economies of scale. Formally, producers of variety ω in industry (i,k) perceive labor productivity as exogenously given by $\varphi_{i,k}(\omega) L_{i,k}^{\gamma_k}$, where $\varphi_{i,k}(\omega)$ is the exogenous component of productivity, $L_{i,k}$ is the total amount of labor employed in industry (i,k), and γ_k is the scale elasticity. The exogenous component of productivity $\varphi_{i,k}(\omega)$ in industry (i,k) is independently drawn across varieties, industries, and countries from a Fréchet distribution given by the cumulative density function $G_{i,k}(\varphi) \equiv \exp\left\{-B_{i,k}\varphi^{-\vartheta_k}\right\}$, where $B_{i,k} > 0$ is the scale parameter, and $\vartheta_k > \sigma_k - 1$ is the shape parameter.

Let $p_{ij,k}(\omega)$ be the price that consumers in country j would be willing to pay for the version of variety ω produced in industry (i,k). Given the import and export taxes, a producer of variety ω in industry (i,k) selling to j chooses its output $y_{ij,k}(\omega)$ so as to maximize its profits given by

$$\pi_{ij,k}(\omega) = \frac{\bar{t}^x_{ij,k} p_{ij,k}(\omega)}{\bar{t}^m_{ij,k}} y_{ij,k}(\omega) - \bar{s}_{i,k} w_i \frac{\overline{\tau}_{ij,k} y_{ij,k}(\omega)}{\varphi_{i,k}(\omega) L_{i,k}^{\gamma_k}},$$

where $\bar{s}_{i,k} \equiv 1 - s_{i,k}$ is one minus i's employment subsidy in sector k, and $\overline{\tau}_{ij,k}$ is the iceberg trade cost. Therefore, under perfect competition, price $p_{ij,k}(\omega)$ is given by

$$p_{ij,k}(\omega) = \frac{\bar{t}_{ij,k} \overline{\tau}_{ij,k} w_i}{\varphi_{i,k}(\omega) L_{i,k}^{\gamma_k}},$$

where $\bar{t}_{ij,k} \equiv \bar{t}^m_{ij,k} \bar{s}_{i,k} / \bar{t}^x_{ij,k}$ is the aggregate policy index of trade from i to j in sector k.

Consumers in each country j shop around the world for the cheapest version of each variety ω in each industry k. Thus, the price paid by consumers in country j is given by $p_{j,k}(\omega) = \min_i \{p_{ij,k}(\omega)\}$. The standard calculations give the consumer price index in country j of goods that country j buys from industry (i,k),

$$P_{ij,k} = \frac{\delta_k^{\text{EK}} \bar{t}_{ij,k} \overline{\tau}_{ij,k} w_i}{B_{i,k}^{\frac{1}{\vartheta_k}} L_{i,k}^{\gamma_k}}, \tag{3}$$

where $\delta_k^{\text{EK}} \equiv \Gamma\left(1 + (1-\sigma_k)/\vartheta_k\right)^{1/(1-\sigma_k)}$ and $\Gamma(\cdot)$ is the Gamma function. The consumer price index in country j associated with all goods consumed in industry k is given by

$$P_{j,k} = \delta_k^{\text{EK}} \cdot \left[\sum_i B_{i,k} \left(\bar{t}_{ij,k} \overline{\tau}_{ij,k} w_i\right)^{-\vartheta_k} L_{i,k}^{\vartheta_k \gamma_k}\right]^{-\frac{1}{\vartheta_k}}, \tag{4}$$

and the expenditure share $\lambda_{ij,k}$ is given by

$$\lambda_{ij,k} = \left(\frac{P_{ij,k}}{P_{j,k}}\right)^{-\vartheta_k} = \frac{B_{i,k} \left(\bar{t}_{ij,k} \overline{\tau}_{ij,k} w_i\right)^{-\vartheta_k} L_{i,k}^{\vartheta_k \gamma_k}}{\sum_l B_{l,k} \left(\bar{t}_{lj,k} \overline{\tau}_{lj,k} w_l\right)^{-\vartheta_k} L_{l,k}^{\vartheta_k \gamma_k}}.$$

The goods market clearing and labor market clearing conditions are the same as in the Armington model described in Appendix 1.1,

$$w_i L_{i,k} = \sum_j \left(\lambda_{ij,k} / \bar{t}_{ij,k}\right) \beta_{j,k} \left(w_j L_j + T_j\right),$$

$$\sum_k L_{i,k} = L_i.$$

1.3 A Generalized Krugman Model

Each country i in each industry k is producing a set of differentiated varieties $\Omega_{i,k}$ that are purchased by all other countries. The number of varieties $M^e_{i,k}$ in the set $\Omega_{i,k}$ is endogenously determined by the free entry condition introduced below. Following Kucheryavyy et al. (2023), we assume that consumers in country j combine varieties originating from industry (i,k) using a Constant Elasticity of Substitution (CES) aggregator with the elasticity of substitution $\sigma_k > 1$ to obtain bilateral country-level consumption indices, $C_{ij,k}$. These bilateral consumption indices are further combined using a CES aggregator with the elasticity of substitution $\eta_k > 1$ to obtain consumption indices $C_{j,k}$.

Let $p_{ij,k}(\omega)$ be the price that consumers in country j pay for variety $\omega \in \Omega_{i,k}$ produced in industry (i,k). Then demand in country j for variety $\omega \in \Omega_{i,k}$ is given by

$$c_{ij,k}(\omega) = \left(\frac{p_{ij,k}(\omega)}{P_{ij,k}}\right)^{-\sigma_k} \frac{X_{ij,k}}{P_{ij,k}},$$

where

$$P_{ij,k} \equiv \left[\int_{\omega\in\Omega_{i,k}} p_{ij,k}(\omega)^{1-\sigma_k}\, d\omega\right]^{\frac{1}{1-\sigma_k}}$$

is the price index of varieties produced in industry (i,k) and sold in country j.

In all countries and industries, producers of varieties engage in monopolistic competition. The efficiency of production in industry (i,k) is the same across all varieties ω and is given by $a_{i,k}$. Given the import and export taxes, a producer of variety ω in industry (i,k) selling to j chooses its price $p_{ij,k}(\omega)$ so as to maximize its profits given by

$$\pi_{ij,k}(\omega) = \frac{\bar{t}^x_{ij,k} p_{ij,k}(\omega)}{\bar{t}^m_{ij,k}} y_{ij,k}(\omega) - \bar{s}_{i,k} w_i \frac{\overline{\tau}_{ij,k} y_{ij,k}(\omega)}{a_{i,k}},$$

subject to the demand $y_{ij,k}(\omega) = c_{ij,k}(\omega)$. Here $\bar{s}_{i,k} \equiv 1 - s_{i,k}$ is one minus i's employment subsidy in sector k, and $\overline{\tau}_{ij,k}$ is the iceberg trade cost. The solution of this problem implies

$$p_{ij,k}(\omega) = \frac{\bar{t}_{ij,k}\overline{\tau}_{ij,k} w_i}{\mu_k a_{i,k}}, \tag{5}$$

where $\mu_k \equiv 1 - 1/\sigma_k \in (0,1)$, and $\bar{t}_{ij,k} \equiv \bar{t}^m_{ij,k}\bar{s}_{i,k}/\bar{t}^x_{ij,k}$ is the aggregate policy index of trade from i to j in sector k. At this price, the maximized profit is

$$\pi_{ij,k}(\omega) = \frac{\bar{s}_{i,k} w_i}{\sigma_k - 1} \cdot \frac{\bar{\tau}_{ij,k} c_{ij,k}(\omega)}{a_{i,k}}.$$

and thus the total profit of country-i's producer of variety ω across all destinations j is given by

$$\sum_j \pi_{ij,k}(\omega) = \frac{\bar{s}_{i,k} w_i}{\sigma_k - 1} \sum_j \frac{\bar{\tau}_{ij,k} c_{ij,k}(\omega)}{a_{i,k}}.$$

Let $f^e_{i,k}$ denote the fixed entry cost in industry (i,k) in terms of i's labor. Assuming free-entry in any industry (i,k), we have $\bar{s}_{i,k} w_i f^e_{i,k} = \sum_j \pi_{ij,k}(\omega)$ for any variety $\omega \in \Omega_{i,k}$, where we took into account i's employment subsidy in sector k. Given the above expression for $\sum_j \pi_{ij,k}(\omega)$, we get

$$\sum_j \frac{\bar{\tau}_{ij,k} c_{ij,k}(\omega)}{a_{i,k}} = (\sigma_k - 1) f^e_{i,k}.$$

Observe that the left-hand side of this equality is the amount labor hired for production in sector (i,k). Using this, the labor market clearing condition in industry (i,k) gives

$$L_{i,k} = \int_{\omega \in \Omega_{i,k}} \left\{ \sum_j \frac{\bar{\tau}_{ij,k} c_{ij,k}(\omega)}{a_{i,k}} + f^e_{i,k} \right\} d\omega = \sigma_k f^e_{i,k} M^e_{i,k}.$$

This implies $M^e_{i,k} = L_{i,k} / \left(\sigma_k f^e_{i,k}\right)$.

Using price (5) and $M^e_{i,k} = L_{i,k} / \left(\sigma_k f^e_{i,k}\right)$, we get

$$P_{ij,k} = \delta^{\mathrm{K}}_k \cdot \frac{\bar{t}_{ij,k} \bar{\tau}_{ij,k} w_i}{a_{i,k} \left(f^e_{i,k}\right)^{-\frac{1}{\sigma_k - 1}} L_{i,k}^{\frac{1}{\sigma_k - 1}}}, \tag{6}$$

where $\delta^{\mathrm{K}}_k \equiv \sigma_k^{\frac{1}{\sigma_k - 1}} \mu_k^{-1}$. This implies that the price index associated with the consumption index in industry (j,k) is given by

$$P_{j,k} = \delta^{\mathrm{K}}_k \cdot \left[\sum_i a_{i,k}^{\eta_k - 1} \left(f^e_{i,k}\right)^{-\frac{\eta_k - 1}{\sigma_k - 1}} \left(\bar{t}_{ij,k} \bar{\tau}_{ij,k} w_i\right)^{-(\eta_k - 1)} L_{i,k}^{\frac{\eta_k - 1}{\sigma_k - 1}} \right]^{\frac{1}{1 - \eta_k}}, \tag{7}$$

and the trade share is given by

$$\lambda_{ij,k} = \left(\frac{P_{ij,k}}{P_{j,k}}\right)^{1-\eta_k} = \frac{a_{i,k}^{\eta_k-1}\left(f_{i,k}^e\right)^{-\frac{\eta_k-1}{\sigma_k-1}}\left(\bar{t}_{ij,k}\overline{\tau}_{ij,k}w_i\right)^{-(\eta_k-1)}L_{i,k}^{\frac{\eta_k-1}{\sigma_k-1}}}{\sum_l a_{l,k}^{\eta_k-1}\left(f_{l,k}^e\right)^{-\frac{\eta_k-1}{\sigma_k-1}}\left(\bar{t}_{lj,k}\overline{\tau}_{lj,k}w_l\right)^{-(\eta_k-1)}L_{l,k}^{\frac{\eta_k-1}{\sigma_k-1}}}.$$

The goods market clearing and labor market clearing conditions are the same as in the Armington model described in Appendix 1.1,

$$w_i L_{i,k} = \sum_j \left(\lambda_{ij,k}/\bar{t}_{ij,k}\right)\beta_{j,k}\left(w_j L_j + T_j\right),$$

$$\sum_k L_{i,k} = L_i.$$

1.4 A Generalized Melitz-Pareto Model with Fixed Marketing Costs Paid in Labor of Source Countries

Each country i in each industry k can produce a set of differentiated varieties $\Omega_{i,k}$. The number of varieties $M_{i,k}^e$ in the set $\Omega_{i,k}$ is endogenously determined by the free entry condition introduced below. Among all varieties from the set $\Omega_{i,k}$, only a subset of varieties $\Omega_{ij,k} \subseteq \Omega_{i,k}$ is available in a particular country j. As in the Krugman model described in Appendix 1.3, demand in any industry (j,k) has a nested CES structure: varieties from a particular country i are combined into a bilateral consumption index $C_{ij,k}$ using a CES aggregator with the elasticity of substitution $\sigma_k > 1$; and bilateral consumption indices are combined into a consumption index $C_{j,k}$ using a CES aggregator with the elasticity of substitution $\eta_k > 1$.

Let $p_{ij,k}(\omega)$ be the price that consumers in country j pay for variety $\omega \in \Omega_{ij,k}$ produced in industry (i,k). Then demand in country j for variety $\omega \in \Omega_{ij,k}$ is given by

$$c_{ij,k}(\omega) = \left(\frac{p_{ij,k}(\omega)}{P_{ij,k}}\right)^{-\sigma_k}\frac{X_{ij,k}}{P_{ij,k}}, \tag{8}$$

where

$$P_{ij,k} \equiv \left[\int_{\omega\in\Omega_{ij,k}} p_{ij,k}(\omega)^{1-\sigma_k}\,d\omega\right]^{\frac{1}{1-\sigma_k}}$$

is the price index of varieties produced in industry (i,k) and sold in country j.

In all countries and industries, producers of varieties engage in monopolistic competition. The efficiency $\varphi_{i,k}(\omega)$ of production of variety $\omega \in \Omega_{i,k}$ from industry (i,k) is independently drawn across firms, industries, and countries from a Pareto distribution given by the cumulative density function $G_{i,k}(\varphi) \equiv 1 - (b_{i,k}/\varphi)^{\theta_k}$ defined for $\varphi \geq b_{i,k}$. As is standard, we assume that $\theta_k > \sigma_k - 1$.

In order to sell a variety in country j, a firm from industry (i,k) needs to pay fixed marketing costs $f_{ij,k}$ in terms of the labor of country i. Given the import and export taxes, a producer of variety ω in industry (i,k) selling to j chooses its price $p_{ij,k}(\omega)$ so as to maximize its profits net of marketing costs,

$$\pi_{ij,k}(\omega) = \frac{\bar{t}^x_{ij,k} p_{ij,k}(\omega)}{\bar{t}^m_{ij,k}} y_{ij,k}(\omega) - \bar{s}_{i,k} w_i \frac{\overline{\tau}_{ij,k} y_{ij,k}(\omega)}{\varphi_{i,k}(\omega)} - \bar{s}_{i,k} w_i f_{ij,k},$$

subject to the demand $y_{ij,k}(\omega) = c_{ij,k}(\omega)$. Here $\bar{s}_{i,k} \equiv 1 - s_{i,k}$ is one minus i's employment subsidy in sector k, and $\overline{\tau}_{ij,k}$ is the iceberg trade cost. The solution of this problem implies

$$p_{ij,k}(\omega) = \frac{\bar{t}_{ij,k} \overline{\tau}_{ij,k} w_i}{\mu_k \varphi_{i,k}(\omega)}, \tag{9}$$

where $\mu_k \equiv 1 - 1/\sigma_k \in (0,1)$, and $\bar{t}_{ij,k} \equiv \bar{t}^m_{ij,k} \bar{s}_{i,k} / \bar{t}^x_{ij,k}$ is the aggregate policy index of trade from i to j in sector k. At this price, the maximized profit is

$$\pi_{ij,k}(\omega) = \frac{\bar{s}_{i,k} \bar{t}^{-1}_{ij,k}}{\sigma_k} \cdot p_{ij,k}(\omega)\, c_{ij,k}(\omega) - \bar{s}_{i,k} w_i f_{ij,k}. \tag{10}$$

Within industry (i,k), firms serving country j are those capable of earning non-negative profits in j. These firms have efficiencies exceeding the threshold $\varphi_{ij,k}$, derived from the condition $\pi_{ij,k}(\omega) = 0$. Substituting (8) and (9) into (10), and solving for $\varphi_{ij,k}$ from $\pi_{ij,k}(\omega) = 0$, gives

$$\varphi_{ij,k} = \mu_k^{-1} \bar{t}_{ij,k}^{\frac{\sigma_k}{\sigma_k - 1}} \frac{\overline{\tau}_{ij,k} w_i}{P_{ij,k}} \left(\frac{X_{ij,k}}{\sigma_k w_i f_{ij,k}} \right)^{-\frac{1}{\sigma_k - 1}}. \tag{11}$$

Given the Pareto distribution of efficiencies $\varphi_{i,k}(\omega)$, we find

$$P_{ij,k} = \mu_k^{-1} \left(\frac{\theta_k - (\sigma_k - 1)}{\theta_k} \right)^{\frac{1}{\sigma_k - 1}} \bar{t}_{ij,k} \overline{\tau}_{ij,k} w_i b_{i,k}^{-\frac{\theta_k}{\sigma_k - 1}} \left[M^e_{i,k} \right]^{-\frac{1}{\sigma_k - 1}} \varphi_{ij,k}^{\frac{\theta_k - (\sigma_k - 1)}{\sigma_k - 1}}. \tag{12}$$

Substituting this into (11) and solving for $\varphi_{ij,k}$, we get

$$\left(\frac{\varphi_{ij,k}}{b_{i,k}}\right)^{-\theta_k} = \frac{\theta_k - (\sigma_k - 1)}{\theta_k \sigma_k} \cdot \frac{1}{M^e_{i,k}} \cdot \frac{X_{ij,k}/\bar{t}_{ij,k}}{w_i f_{ij,k}}.$$

Using this, we can find the number of varieties in the set $\Omega_{ij,k}$,

$$M_{ij,k} = \left(1 - G_{i,k}\left(\varphi_{ij,k}\right)\right) M^e_{i,k} = \left(\frac{\varphi_{ij,k}}{b_{i,k}}\right)^{-\theta_k} M^e_{i,k} = \frac{\theta_k - (\sigma_k - 1)}{\theta_k \sigma_k} \cdot \frac{X_{ij,k}/\bar{t}_{ij,k}}{w_i f_{ij,k}}. \tag{13}$$

Noting that $p_{ij,k}(\omega)\, c_{ij,k}(\omega)$ in the expression (10) for $\pi_{ij,k}(\omega)$ is the expenditure of country j on variety $\omega \in \Omega_{ij,k}$, we can integrate the expression for $\pi_{ij,k}(\omega)$ over all varieties $\omega \in \Omega_{ij,k}$ to get

$$\int_{\omega \in \Omega_{ij,k}} \pi_{ij,k}(\omega)\, d\omega = \frac{\bar{s}_{i,k} X_{ij,k}/\bar{t}_{ij,k}}{\sigma_k} - \bar{s}_{i,k} w_i f_{ij,k} M_{ij,k}.$$

Substituting expression (13) for $M_{ij,k}$ into this, we get

$$\int_{\omega \in \Omega_{ij,k}} \pi_{ij,k}(\omega)\, d\omega = \frac{\sigma_k - 1}{\theta_k \sigma_k} \cdot \bar{s}_{i,k} X_{ij,k}/\bar{t}_{ij,k}. \tag{14}$$

As is standard, we assume that in order to enter industry (i,k), producer of variety ω pays entry cost $f^e_{i,k}$ in terms of country i's labor. Under the free entry condition, we get

$$\bar{s}_{i,k} w_i f^e_{i,k} M^e_{i,k} = \sum_j \int_{\omega \in \Omega_{ij,k}} \pi_{ij,k}(\omega)\, d\omega.$$

Substituting (14) into this, we get

$$M^e_{i,k} f^e_{i,k} = \frac{\sigma_k - 1}{\theta_k \sigma_k} \cdot \sum_j \frac{X_{ij,k}/\bar{t}_{ij,k}}{w_i}, \tag{15}$$

where the left-hand side of this expression is the amount of labor used for entry in industry (i,k).

Next, the amount of labor employed in production of variety $\omega \in \Omega_{ij,k}$ pruchased by country j, is given by $\ell_{ij,k}(\omega) = \bar{\tau}_{ij,k} y_{ij,k}(\omega) / \varphi_{i,k}(\omega)$. Multiplying the right-hand side of this expression by $p_{ij,k}(\omega)/p_{ij,k}(\omega)$ and using (9) in the denominator, we get

$$\ell_{ij,k}(\omega) = \frac{\sigma_k - 1}{\sigma_k} \cdot \frac{p_{ij,k}(\omega)\, y_{ij,k}(\omega)/\bar{t}_{ij,k}}{w_i}.$$

Integrating this over all varieties $\omega \in \Omega_{ij,k}$, we get

$$\int_{\omega \in \Omega_{ij,k}} \ell_{ij,k}(\omega)\, d\omega = \frac{\sigma_k - 1}{\sigma_k} \cdot \frac{X_{ij,k}/\bar{t}_{ij,k}}{w_i}. \tag{16}$$

In addition to the production labor, country i employs labor to pay marketing costs in each country j. The amount of labor employed for these marketing costs is given by

$$M_{ij,k} f_{ij,k} = \frac{\theta_k - (\sigma_k - 1)}{\theta_k \sigma_k} \cdot \frac{X_{ij,k}/\bar{t}_{ij,k}}{w_i}, \tag{17}$$

where we used expression (13) for $M_{ij,k}$.

Let $L_{i,k}$ be the total amount of labor employed in industry (i,k). Adding up all uses of this labor given by (15), (16), and (17), we get the goods market clearing condition in industry (i,k),

$$w_i L_{i,k} = \sum_j X_{ij,k}/\bar{t}_{ij,k}.$$

Substituting $\sum_j X_{ij,k}/\bar{t}_{ij,k}$ from here into (15), we get

$$M^e_{i,k} = \frac{\sigma_k - 1}{\theta_k \sigma_k} \cdot \frac{L_{i,k}}{f^e_{i,k}}. \tag{18}$$

The labor market clearing condition is the same as in all other models, $\sum_k L_{i,k} = L_i$, and we only need to use (12) to derive expressions for $P_{ij,k}$ and $P_{j,k}$ without $\varphi_{ij,k}$, and then use these expressions to derive trade shares $\lambda_{ij,k}$. For that, substitute expressions (11) and (18) for $\varphi_{ij,k}$ and $M^e_{i,k}$ into the expression (12) for $P_{ij,k}$, and solve for $P_{ij,k}$ from the resulting expression to get

$$P_{ij,k} = \delta^{\mathrm{M}}_{j,k} X^{1-\iota_k}_{ij,k} \left(\bar{t}^{\iota_k}_{ij,k} \tau_{ij,k} w^{\iota_k}_i / A_{i,k} \right) L^{-\frac{1}{\theta_k}}_{i,k}, \tag{19}$$

where $\delta^{\mathrm{M}}_{j,k} \equiv \left(\frac{\theta_k}{\sigma_k - 1} - 1 \right)^{\frac{1}{\theta_k}} \sigma_k^{\frac{1}{\sigma_k - 1}} \mu_k^{-1} f^{\iota_k - 1}_{jj,k}$, $\iota_k \equiv 1 + \frac{1}{\sigma_k - 1} - \frac{1}{\theta_k} > 1$, and

$$A_{i,k} \equiv \left[f^e_{i,k} \right]^{-\frac{1}{\theta_k}} b_{i,k},$$
$$\tau_{ij,k} \equiv \left(f_{ij,k}/f_{jj,k} \right)^{\iota_k - 1} \bar{\tau}_{ij,k}.$$

Expression (19) for $P_{ij,k}$ corresponds to expression (20) in the main text. We also have $\eta_k - 1 = (1 + \iota_k [\xi_k \theta_k]) [\xi_k \theta_k]^{-1} - 1$, where

$$\xi_k \equiv \left[1 + \left(\frac{1}{\eta_k - 1} - \frac{1}{\sigma_k - 1}\right)\theta_k\right]^{-1} \in [0,1].$$

Given the definitions in Table 1 in the main text, this implies $\eta_k - 1 = (1 + \zeta_k)/\varepsilon_k - 1$. Therefore, $P_{j,k} = \left(\sum_i P_{ij,k}^{1-\eta_k}\right)^{1/(1-\eta_k)}$ can be written as (21) in the main text.

Substituting $X_{ij,k} = \left(P_{ij,k}/P_{j,k}\right)^{1-\eta_k} X_{j,k}$ into (19), and solving for $P_{ij,k}$, we get

$$P_{ij,k}^{1-\eta_k} = \left[\delta_{j,k}^{\mathrm{M}}\right]^{-\theta_k \xi_k} \left(P_{j,k}^{\eta_k - 1} X_{j,k}\right)^{(\iota_k - 1)\theta_k \xi_k} \left(\bar{t}_{ij,k}^{\iota_k} \tau_{ij,k} w_i^{\iota_k} / A_{i,k}\right)^{-\theta_k \xi_k} L_{i,k}^{\xi_k},$$

Substituting the above expression for $P_{ij,k}^{1-\eta_k}$ into $P_{j,k}^{1-\eta_k} = \sum_i P_{ij,k}^{1-\eta_k}$, and solving for $P_{j,k}$, we get

$$P_{j,k} = \delta_{j,k}^{\mathrm{M}} X_{j,k}^{1-\iota_k} \left[\sum_i \left(\bar{t}_{ij,k}^{\iota_k} \tau_{ij,k} w_i^{\iota_k} / A_{i,k}\right)^{-\theta_k \xi_k} L_{i,k}^{\xi_k}\right]^{-\frac{1}{\theta_k \xi_k}}. \tag{20}$$

Finally, trade shares are given by

$$\lambda_{ij,k} = \left(\frac{P_{ij,k}}{P_{j,k}}\right)^{1-\eta_k} = \frac{\left(\bar{t}_{ij,k}^{\iota_k} \tau_{ij,k} w_i^{\iota_k} / A_{i,k}\right)^{-\theta_k \xi_k} L_{i,k}^{\xi_k}}{\sum_l \left(\bar{t}_{lj,k}^{\iota_k} \tau_{lj,k} w_l^{\iota_k} / A_{l,k}\right)^{-\theta_k \xi_k} L_{l,k}^{\xi_k}}.$$

1.5 A Generalized Melitz-Pareto Model with Fixed Marketing Costs Paid in Labor of Destination Countries

The setup is almost the same as in the Melitz-Pareto model described in Appendix 1.4 with fixed costs paid in labor of source countries. The only difference is that fixed marketing costs are paid in labor of destination countries. However, this difference has non-trivial consequences for the labor market clearing condition.

The profit of producer of variety ω from industry (i,k) selling its variety in country j is now given by

$$\pi_{ij,k}(\omega) = \frac{\bar{t}_{ij,k}^{x} p_{ij,k}(\omega)}{\bar{t}_{ij,k}^{m}} y_{ij,k}(\omega) - \bar{s}_{i,k} w_i \frac{\overline{\tau}_{ij,k} y_{ij,k}(\omega)}{\varphi_{i,k}(\omega)} - \bar{s}_{j,k} w_j f_{ij,k}.$$

Here, the last term involves the wage of country j and, for consistency with other micro-foundations, one minus j's employment subsidy in sector k, denoted as $\bar{s}_{j,k} \equiv 1 - s_{j,k}$, ensuring that the employment subsidy for all labor from one country is funded by that respective country.[1] The solution of this problem implies the same price as in the Melitz-source model,

$$p_{ij,k}(\omega) = \frac{\bar{t}_{ij,k}\bar{\tau}_{ij,k}w_i}{\mu_k \varphi_{i,k}(\omega)}, \tag{21}$$

where $\mu_k \equiv 1 - 1/\sigma_k \in (0,1)$ and $\bar{t}_{ij,k} \equiv \bar{t}^m_{ij,k}\bar{s}_{i,k}/\bar{t}^x_{ij,k}$. The maximized profit is

$$\pi_{ij,k}(\omega) = \frac{\bar{s}_{i,k}\bar{t}^{-1}_{ij,k}}{\sigma_k} \cdot p_{ij,k}(\omega)\, c_{ij,k}(\omega) - \bar{s}_{j,k}w_j f_{ij,k}. \tag{22}$$

Solving for the cutoff efficiency $\varphi_{ij,k}$ from $\pi_{ij,k}(\omega) = 0$, we get

$$\varphi_{ij,k} = \mu_k^{-1}\left(\bar{s}_{j,k}/\bar{s}_{i,k}\right)^{\frac{1}{\sigma_k-1}} \bar{t}_{ij,k}^{\frac{\sigma_k}{\sigma_k-1}} \frac{\bar{\tau}_{ij,k}w_i}{P_{ij,k}} \left(\frac{X_{ij,k}}{\sigma_k w_j f_{ij,k}}\right)^{-\frac{1}{\sigma_k-1}}. \tag{23}$$

The bilateral price index $P_{ij,k}$ has the same expression as in the Melitz-Pareto-source model (see (12)),

$$P_{ij,k} = \mu_k^{-1}\left(\frac{\theta_k - (\sigma_k - 1)}{\theta_k}\right)^{\frac{1}{\sigma_k-1}} \bar{t}_{ij,k}\bar{\tau}_{ij,k}w_i b_{i,k}^{-\frac{\theta_k}{\sigma_k-1}} \left[M^e_{i,k}\right]^{-\frac{1}{\sigma_k-1}} \varphi_{ij,k}^{\frac{\theta_k-(\sigma_k-1)}{\sigma_k-1}}. \tag{24}$$

Substituting this into (23) and solving for $\varphi_{ij,k}$, we get

$$\left(\frac{\varphi_{ij,k}}{b_{i,k}}\right)^{-\theta_k} = \frac{\theta_k - (\sigma_k - 1)}{\theta_k\sigma_k} \cdot \frac{1}{M^e_{i,k}} \cdot \frac{\left(\bar{s}_{i,k}/\bar{s}_{j,k}\right) X_{ij,k}/\bar{t}_{ij,k}}{w_j f_{ij,k}}.$$

Using this, we can find the number of varieties in the set $\Omega_{ij,k}$,

$$M_{ij,k} = \left(\frac{\varphi_{ij,k}}{b_{i,k}}\right)^{-\theta_k} M^e_{i,k} = \frac{\theta_k - (\sigma_k - 1)}{\theta_k\sigma_k} \cdot \frac{\left(\bar{s}_{i,k}/\bar{s}_{j,k}\right) X_{ij,k}/\bar{t}_{ij,k}}{w_j f_{ij,k}}. \tag{25}$$

Integrating the expression for $\pi_{ij,k}(\omega)$ over all varieties $\omega \in \Omega_{ij,k}$, we get

$$\int_{\omega\in\Omega_{ij,k}} \pi_{ij,k}(\omega)\, d\omega = \frac{\bar{s}_{i,k}X_{ij,k}/\bar{t}_{ij,k}}{\sigma_k} - \bar{s}_{j,k}w_j f_{ij,k}M_{ij,k}.$$

[1] Alternatively, employment subsidies for marketing costs can be paid by the source country. In this case, the term $\bar{s}_{j,k}w_j f_{ij,k}$ in the expression for profits will be replaced by $\bar{s}_{i,k}w_j f_{ij,k}$. As we do not derive the first-best policy for the Melitz-Pareto-destination model in this paper, we do not know which policy instruments are necessary to achieve the optimal outcome. Consequently, neither specification for the employment subsidies is inherently superior to the other.

Substituting expression (25) for $M_{ij,k}$ into this, we get the same expression as in the Melitz-Pareto-source model (see (14)),

$$\int_{\omega\in\Omega_{ij,k}} \pi_{ij,k}(\omega)\, d\omega = \frac{\sigma_k - 1}{\theta_k \sigma_k} \cdot \bar{s}_{i,k} X_{ij,k} / \bar{t}_{ij,k}. \tag{26}$$

The free-entry condition in industry (i,k) is given by

$$\bar{s}_{i,k} w_i f^e_{i,k} M^e_{i,k} = \sum_j \int_{\omega\in\Omega_{ij,k}} \pi_{ij,k}(\omega)\, d\omega.$$

Substituting (26) into this, we get the amount of labor used for entry in industry (i,k),

$$M^e_{i,k} f^e_{i,k} = \frac{\sigma_k - 1}{\theta_k \sigma_k} \cdot \sum_j \frac{X_{ij,k}/\bar{t}_{ij,k}}{w_i}, \tag{27}$$

which is the same expression as in the Melitz-Pareto-source model (see (15)).

The amount of labor employed in production of all varieties $\omega \in \Omega_{ij,k}$ produced in industry (i,k) and pruchased by country j, is also given by the same expression as in the Melitz-Pareto-source model (see (16)),

$$\int_{\omega\in\Omega_{ij,k}} \ell_{ij,k}(\omega)\, d\omega = \frac{\sigma_k - 1}{\sigma_k} \cdot \frac{X_{ij,k}/\bar{t}_{ij,k}}{w_i}. \tag{28}$$

Finally, the amount of country-i's labor employed to pay marketing costs for imports from country j is given by

$$M_{ji,k} f_{ji,k} = \frac{\nu_k \cdot \left(\bar{s}_{j,k}/\bar{s}_{i,k}\right) X_{ji,k}/\bar{t}_{ji,k}}{w_i}, \tag{29}$$

where $\nu_k \equiv \frac{\theta_k - (\sigma_k - 1)}{\theta_k \sigma_k}$, which is different from the corresponding expression in the Melitz-Pareto-source model (see (17)).

Let $L^{\text{pe}}_{i,k}$ be the amount of labor employed in industry (i,k) for production and entry. Using (27) and (28), we get

$$w_i L^{\text{pe}}_{i,k} = (1 - \nu_k) \sum_j X_{ij,k}/\bar{t}_{ij,k},$$

which could potentially serve as the counterpart of the goods market clearing condition in the generalized gravity model of Section 5 of the main text.

The labor market clearing condition is given by

$$\sum_k L^{\text{pe}}_{i,k} + \sum_k \sum_j \frac{\nu_k \left(\bar{s}_{j,k}/\bar{s}_{i,k}\right) X_{ji,k}/\bar{t}_{ji,k}}{w_i} = L_i,$$

which in general is different from the other models. The issue is that this condition is directly affected by the policy instruments. Without policy instruments ($\bar{t}^m_{ij,k} = 1$, $\bar{t}^x_{ij,k} = 1$, and $\bar{s}_{i,k} = 1$, for all i, j, and k), we have the same setup as in Kucheryavyy et al. (2023). In this case (assuming that expenditure of country i is equal to its income, $X_i = w_i L_i$), we can write the left-hand side of the labor market clearing condition as

$$\sum_k L^{\text{pe}}_{i,k} + \sum_k \frac{\nu_k}{w_i} \sum_j \lambda_{ji,k} \beta_{i,k} w_i L_i = \sum_k L^{\text{pe}}_{i,k} + \left(\sum_k \nu_k \beta_{i,k} \right) L_i,$$

and thus the labor market clearing condition can be written as

$$\sum_k L^{\text{pe}}_{i,k} = \left(1 - \sum_k \nu_k \beta_{i,k} \right) L_i.$$

This differs from the labor market clearing condition in the other models just by a constant on the right-hand side. This is inconsequential for equilibrium and welfare analysis. However, in the presence of the policy instruments, the different form of the labor market clearing condition in the Melitz-Pareto-destination model potentially has nontririval consequences for equilibrium and welfare analysis as well as for policy analysis.

Another nontrivial feature of the multi-sector Melitz-Pareto-destination model, distinguishing it from the other four models, is its reliance on a current account balance condition instead of the trade balance condition. This means that trade balance is not necessarily maintained in the Melitz-Pareto-destination model. Formally, the current account balance condition for country i can be expressed as

$$\sum_k \sum_j \frac{X_{ij,k}}{\bar{t}^m_{ij,k}} + \sum_k \sum_j \frac{\nu_k \left(\bar{s}_{j,k}/\bar{s}_{i,k}\right) X_{ji,k}}{\bar{t}_{ji,k}} = \sum_k \sum_j \frac{X_{ji,k}}{\bar{t}^m_{ji,k}} + \sum_k \sum_j \frac{\left(\bar{s}_{i,k}/\bar{s}_{j,k}\right) \nu_k X_{ij,k}}{\bar{t}_{ij,k}}. \tag{30}$$

On the income side, the left-hand side of (30) consists of country-i's exports and the income earned by country-i's labor from abroad for employment in marketing (see expression (29)). On the expenditure side, the right-hand side of (30) comprises country-i's imports and the payments made by country i to foreign labor. As evident from expression (29), the value of country-i's exports does not necessarily equal the value of imports, $\sum_k \sum_j X_{ij,k}/\bar{t}^m_{ij,k} \neq \sum_k \sum_j X_{ji,k}/\bar{t}^m_{ji,k}$.

2 Microfoundations of the Single-Sector Gravity Model

With the exception of the Melitz-Pareto-destination model, the four setups presented in Appendix 1 result in the same (up to relabeling) system of equilibrium equations described in Section 5. This system of equations is given by the trade shares (16), tax revenues (18), goods market clearing conditions (17), and labor market clearing conditions $\sum_k L_{i,k} = L_i$. Transitioning to the single-sector version of this system, as presented in Section 2 with ad-valorem import tariffs as the sole policy instrument, is straightforward. Here, the labor market clearing condition becomes unnecessary, and the labor endowment L_i replaces labor demand $L_{i,k}$ in both the goods market clearing condition and the expression for trade shares. Removing the sector index k and substituting $\bar{t}_{ij} = \bar{t}^m_{ij,k}$, $\bar{t}^x_{ji,k} = 1$, and $\bar{s}_{j,k} = 1$, we derive the single-sector system of equilibrium conditions:

$$w_i L_i = \sum_j \left(\lambda_{ij}/\bar{t}_{ij}\right)\left(w_j L_j + T_j\right), \tag{31}$$

$$\lambda_{ij} = \frac{\bar{t}_{ij}^{-\zeta}\left(\tau_{ij}/A_i\right)^{-\varepsilon} w_i^{-\zeta} L_i^{\alpha}}{\sum_l \bar{t}_{lj}^{-\zeta}\left(\tau_{lj}/A_l\right)^{-\varepsilon} w_l^{-\zeta} L_l^{\alpha}}, \tag{32}$$

$$T_j = \sum_i \frac{\bar{t}_{ij}-1}{\bar{t}_{ij}}\lambda_{ij}\left(w_j L_j + T_j\right). \tag{33}$$

Expression (32) for trade shares is already in the form as presentented in Section 2 (see expression (1) there), with the difference that the trade elasticity with respect to wages in (32) is ζ instead of ρ. This is because $\rho \neq \zeta$ only in the Melitz-Pareto-destination model, which we describe below.

Solving for T_j from (33), we obtain

$$T_j = \left(\frac{1}{\sum_i \lambda_{ij}/\bar{t}_{ij}} - 1\right) w_j L_j, \tag{34}$$

where we used $\sum_i \lambda_{ij} = 1$. Substituting this into the goods market clearing condition (31), we have

$$w_i L_i = \sum_j \Lambda_{ij} w_j L_j,$$

where

$$\Lambda_{ij} \equiv \frac{\lambda_{ij}/\bar{t}_{ij}}{\sum_l \lambda_{lj}/\bar{t}_{lj}}.$$

These are the same expressions as in Section 2 (see expressions (2) and (3) there).

We now turn to the Melitz-Pareto-destination model and show that its single-sector version yields the same (up to relabeling) system of equations as the one presented in Section 2.

2.1 A Generalized Melitz-Pareto Model with Fixed Marketing Costs Paid in Labor of Destination Countries

Continuing from where we left off in Appendix 1.5, the single-sector version of the current account balance condition (30) for country i is given by

$$\sum_j \frac{X_{ij}}{\bar{t}_{ij}} + \nu \sum_j \frac{X_{ji}}{\bar{t}_{ji}} = \sum_j \frac{X_{ji}}{\bar{t}_{ji}} + \nu \sum_j \frac{X_{ij}}{\bar{t}_{ij}},$$

with $\nu \equiv \frac{\theta-(\sigma-1)}{\theta\sigma}$, which is equivalent to the trade balance condition $\sum_j X_{ji}/\bar{t}_{ji} = \sum_j X_{ij}/\bar{t}_{ij}$.

Next, expression (29) for country-i's labor employed to pay marketing costs for imports from country j is given by $M_{ji}f_{ji} = \nu\left(X_{ji}/\bar{t}_{ji}\right)/w_i$. Summing across all source countries j and using the trade balance condition, we obtain

$$\sum_j M_{ji}f_{ji} = \frac{\nu \sum_j X_{ij}/\bar{t}_{ij}}{w_i}.$$

This is the key step that allows us to derive the system of equilibrium equations for the Melitz-Pareto-destination model, aligning it with the system of equations in the single-sector gravity model of Section 2 of the main text. Subsequent steps follow from this derivation.

The single-sector versions of expressions (27) and (28) for labor used for production and entry are given by

$$M_i^e f_i^e = \frac{\sigma-1}{\theta\sigma} \cdot \sum_j \frac{X_{ij}/\bar{t}_{ij}}{w_i} \qquad (35)$$

$$\int_{\omega\in\Omega_{ij,k}} \ell_{ij}(\omega)\, d\omega = \frac{\sigma-1}{\sigma} \cdot \frac{X_{ij}/\bar{t}_{ij}}{w_i}.$$

Adding up all uses of labor, we get the goods/labor market clearing condition,

$$w_i L_i = \sum_j X_{ij} / \bar{t}_{ij}.$$

Given that $X_{ij} = \lambda_{ij} \left(w_j L_j + T_j \right)$, this is the same condition as (31).

Next, substituting $\sum_j X_{ij} / \bar{t}_{ij} = w_i L_i$ into (35), we get

$$M_i^e = \frac{\sigma - 1}{\theta \sigma} \cdot \frac{L_i}{f_i^e}.$$

Substituting this and the single-sector version of expression (23) for φ_{ij} into the single-sector version of expression (24) for P_{ij}, using $X_{ij} = \left(P_{ij} / P_j \right)^{1-\eta} X_j$, and solving for P_{ij} from the resulting expression, we get

$$P_{ij}^{1-\eta} = \left[\delta_j^{\mathrm{M}} \right]^{-\theta \xi} \left(\frac{P_j^{\eta - 1} X_j}{w_j} \right)^{(\iota - 1) \theta \xi} \left(\bar{t}_{ij}^{\iota} \tau_{ij} w_i / A_i \right)^{-\theta \xi} L_i^{\xi},$$

where $\delta_j^{\mathrm{M}} \equiv \left(\frac{\theta}{\sigma - 1} - 1 \right)^{\frac{1}{\theta}} \sigma^{\frac{1}{\sigma - 1}} \mu^{-1} f_{jj}^{\iota - 1}$, $A_i \equiv b_i \left[f_i^e \right]^{-\frac{1}{\theta}}$, and $\tau_{ij} \equiv \left(f_{ij} / f_{jj} \right)^{\iota - 1} \overline{\tau}_{ij}$, with

$$\xi \equiv \left[1 + \left(\frac{1}{\eta - 1} - \frac{1}{\sigma - 1} \right) \theta \right]^{-1} \in [0, 1],$$
$$\iota \equiv 1 + \frac{1}{\sigma - 1} - \frac{1}{\theta} > 1.$$

Substituting the above expression for $P_{ij}^{1-\eta}$ into $P_j^{1-\eta} = \sum_i P_{ij}^{1-\eta}$, and solving for P_j, we get

$$P_j = \delta_j^{\mathrm{M}} \left(x_j L_j \right)^{-(\iota - 1)} \left[\sum_i \left(\bar{t}_{ij}^{\iota} \tau_{ij} w_i / A_i \right)^{-\theta \xi} L_i^{\xi} \right]^{-\frac{1}{\theta \xi}}, \tag{36}$$

where $x_j \equiv \left(w_j L_j + T_j \right) / \left(w_j L_j \right)$.

Finally, trade shares are given by

$$\lambda_{ij} = \left(\frac{P_{ij}}{P_j} \right)^{1-\eta} = \frac{\left(\bar{t}_{ij}^{\iota} \tau_{ij} w_i / A_i \right)^{-\theta \xi} L_i^{\xi}}{\sum_l \left(\bar{t}_{lj}^{\iota} \tau_{lj} w_l / A_l \right)^{-\theta \xi} L_l^{\xi}},$$

which corresponds to expression (1) in Section 2.

2.2 Price Indices and Welfare

The country-level price index encompassing all five microfoundations in the single-sector case can be written as

$$P_j = \delta_j w_j^{-\left(\frac{\rho}{\varepsilon}-1\right)} \left(x_j L_j\right)^{-\left(\frac{\zeta}{\varepsilon}-1\right)} \left[\sum_i \bar{t}_{ij}^{-\zeta} \left(\tau_{ij}/A_i\right)^{-\varepsilon} w_i^{-\rho} L_i^{\alpha}\right]^{-\frac{1}{\varepsilon}}, \qquad (37)$$

where $x_j \equiv \left(w_j L_j + T_j\right) / \left(w_j L_j\right)$. Indeed, in the multi-sector versions of the Armington, Eaton-Kortum, and Krugman models, the country-level price indices are given by expressions (2), (4), and (7) derived in Appendices 1.1, 1.2, and 1.3, correspondingly. Their single-sector version is given by expression (37) for $\zeta = \rho = \varepsilon$. In the multi-sector version of the Melitz-Pareto-source model, the country-level price index is given by (20) derived in Appendix 1.4. Its single-sector version is given by expression (37) for $\zeta = \rho$. Finally, setting $\rho = \varepsilon$ in (37) yields expression (36) for P_j in the Melitz-Pareto-destination model, as derived in Appendix 2.1.

Next, expression (34) for T_j is valid for all five models in the single-sector case. Substituting this expression into $x_j = \left(w_j L_j + T_j\right) / \left(w_j L_j\right)$, we get $x_j = \left(\sum_i \lambda_{ij}/\bar{t}_{ij}\right)^{-1}$, and thus (37) can be written as

$$P_j = \delta_j w_j^{-\left(\frac{\rho}{\varepsilon}-1\right)} \left(\sum_i \frac{\lambda_{ij}/\bar{t}_{ij}}{L_j}\right)^{\frac{\zeta}{\varepsilon}-1} \left[\sum_i \bar{t}_{ij}^{-\zeta} \left(\tau_{ij}/A_i\right)^{-\varepsilon} w_i^{-\rho} L_i^{\alpha}\right]^{-\frac{1}{\varepsilon}}.$$

This is the same expression as expression (4) in Section 2 of the main text.

Finally, for all five microfoundations, welfare in country j is given by $W_j \equiv \left(w_j + T_j/L_j\right) \Big/ P_j$. Again, using expression (34) for T_j we get

$$W_j \equiv \frac{1}{\sum_i \lambda_{ij}/\bar{t}_{ij}} \cdot \frac{w_j}{P_j}.$$

3 Proof of Proposition 1 from the Main Text (Existence and Uniqueness of a Single-Sector Equilibrium)

The equilibrium system can be rewritten as

$$z_i\left(w\right) = 0, \quad i = 0, 1, ..., N,$$

where $z_i(w)$ is country i's excess labor demand function implied by the labor market clearing condition,

$$z_i(w) \equiv \left(\sum_j \Lambda_{ij}(w) w_j L_j - w_i L_i \right) / w_i.$$

We first show existence of a wage vector $w \gg 0$ such that $z_i(w) = 0$ for all i. For that we show that the excess labor demand function $z_i(w) \equiv (z_0(w), z_1(w), ..., z_N(w))$ defined for all $w \gg 0$ satisfies the properties in Proposition 17.B.2 of Mas-Colell et al. (1995): (i) $z(w)$ is continuous; (ii) $z(w)$ is homogeneous of degree zero; (iii) $w \cdot z(w) = 0$ for all w (Walras' law); (iv) there exists $s > 0$ such that $z_i(w) > -s$ for all i and for all w; (v) if $w^m \to w^0$, where $w^0 \neq 0$ and $w_i^0 = 0$ for some i, then $\max\{z_0(w^m), z_1(w^m), \ldots, z_N(w^m)\} \to \infty$.

Properties (i) to (iii) are obvious, while property (iv) is obtained by letting s be larger than $\max_i L_i$. For property (v), consider a wage vector w^0, where $w_i^0 = 0$ for some i, and $w_l^0 > 0$ for all $l \neq i$. Then for any j,

$$\begin{aligned}
\lim_{w^m \to w^0} \Lambda_{ij}(w^m) &= \frac{\bar{t}_{ij}^{-(\zeta+1)} (\tau_{ij}/A_i)^{-\varepsilon} (w_i^0)^{-\rho} L_i^{\alpha}}{\bar{t}_{ij}^{-(\zeta+1)} (\tau_{ij}/A_i)^{-\varepsilon} (w_i^0)^{-\rho} L_i^{\alpha} + \sum_{l \neq i} \bar{t}_{lj}^{-(\zeta+1)} (\tau_{lj}/A_l)^{-\varepsilon} (w_l^0)^{-\rho} L_l^{\alpha}} \\
&= \frac{1}{1 + \bar{t}_{ij}^{\zeta+1} (\tau_{ij}/A_i)^{\varepsilon} (w_i^0)^{\rho} L_i^{-\alpha} \sum_{l \neq i} \bar{t}_{lj}^{-(\zeta+1)} (\tau_{lj}/A_l)^{-\varepsilon} (w_l^0)^{-\rho} L_l^{\alpha}} \\
&= 1.
\end{aligned}$$

This implies that

$$\begin{aligned}
\lim_{w^m \to w^0} z_i(w^m) &= \underbrace{\Lambda_{ii}\left(w^0\right)}_{=1} L_i + \sum_{j \neq i} \underbrace{\Lambda_{ij}\left(w^0\right)}_{=1} w_j^0 L_j / w_i^0 - L_i \\
&= \sum_{j \neq i} w_j^0 L_j / w_i^0 \\
&= \infty,
\end{aligned}$$

which verifies property (v). Since Proposition 17.B.2 constitutes a set of sufficient conditions for Proposition 17.C.1 of Mas-Colell et al. (1995), there exists an equilibrium wage vector $w \gg 0$ such that $z(w) = 0$.

We next show uniqueness. For that we show that $z(w)$ satisfies the gross substitutes property in Definition 17.F.2 of Mas-Colell et al. (1995): if w' and w are such that $w_l' > w_l$

for some l and $w'_i = w_i$ for all $i \neq l$, then $z_i(w') > z_i(w)$ for all $i \neq l$. With differentiability, $\partial z_i(w)/\partial w_l > 0$ for all i and $l \neq i$ ensures the gross substitutes property. Direct calculation gives

$$\begin{aligned} \partial z_i/\partial w_l &= \left\{ \left[\Lambda_{il} + \left(\partial \Lambda_{il}/\partial w_l\right) w_l\right] L_l + \sum_{j \neq l} \left(\partial \Lambda_{ij}/\partial w_l\right) w_j L_j \right\} / w_i \\ &= \left[\Lambda_{il} L_l + \sum_{j} \left(\partial \Lambda_{ij}/\partial w_l\right) w_j L_j \right] / w_i. \end{aligned}$$

Noting that w_l for $l \neq i$ appears only in the denominator of

$$\Lambda_{ij} = \frac{\bar{t}_{ij}^{-(\zeta+1)} \left(\tau_{ij}/A_i\right)^{-\varepsilon} w_i^{-\rho} L_i^{\alpha}}{\sum_l \bar{t}_{lj}^{-(\zeta+1)} \left(\tau_{lj}/A_l\right)^{-\varepsilon} w_l^{-\rho} L_l^{\alpha}},$$

we immediately have $\partial \Lambda_{ij}/\partial w_l > 0$ for all i, j, and $l \neq i$. This implies that $\partial z_i(w)/\partial w_l > 0$ for all i, j, and $l \neq i$. Since the gross substitutes property is a sufficient condition for Proposition 17.F.3 of Mas-Colell et al. (1995), there exists a unique equilibrium wage vector $w \gg 0$ such that $z(w) = 0$.

4 Proof of Proposition 2 from the Main Text (Convergence in the Single-Sector Case)

As shown in Proposition 1 in the main text, for any $n > 0$ there exists a unique equilibrium. This implies that there is a well-defined sequence of equilibria associated with a sequence of n as $n \to 0$.

For compactness of exposition, let us define

$$\begin{aligned} a_{ij} &\equiv A_i^{\varepsilon} \bar{t}_{ij}^{-(\zeta+1)} \tau_{ij}^{-\varepsilon} L_i^{\alpha} \quad i, j = 1, ..., N; \\ a_{0j} &\equiv A_0^{\varepsilon} \bar{t}_{0j}^{-(\zeta+1)} \widetilde{\tau}_{0j}^{-\varepsilon} \widetilde{L}_0^{\alpha}, \quad j = 1, ..., N; \\ a_{i0} &\equiv A_i^{\varepsilon} \bar{t}_{i0}^{-(\zeta+1)} \widetilde{\tau}_{i0}^{-\varepsilon} L_i^{\alpha}, \quad i = 1, ..., N; \\ a_{00} &\equiv A_0^{\varepsilon} \widetilde{L}_0^{\alpha}. \end{aligned}$$

With these definitions, expressions (59)-(62) from the main text can be written as

$$\Lambda_{ij} = \frac{a_{ij} w_i^{-\rho}}{n a_{0j} w_0^{-\rho} + \sum_{l=1}^{N} a_{lj} w_l^{-\rho}}, \quad i,j = 1,...,N;$$
$$\frac{\Lambda_{0j}}{n} = \frac{a_{0j} w_0^{-\rho}}{n a_{0j} w_0^{-\rho} + \sum_{l=1}^{N} a_{lj} w_l^{-\rho}}, \quad j = 1,...,N;$$
$$\Lambda_{i0} = \frac{a_{i0} w_i^{-\rho}}{a_{00} w_0^{-\rho} + \sum_{l=1}^{N} a_{l0} w_l^{-\rho}}, \quad i = 1,...,N;$$
$$\Lambda_{00} = \frac{a_{00} w_0^{-\rho}}{a_{00} w_0^{-\rho} + \sum_{l=1}^{N} a_{l0} w_l^{-\rho}}.$$

Consider any sequence of positive numbers $\{n_r\}_{r=1}^{\infty}$ such that $\lim_{r\to\infty} n_r = 0$. Let us use superscript "(n_r)" to denote the values of all variables in equilibrum of our gravity model with $n = n_r$. In particular, let $\boldsymbol{w}^{(n_r)} \equiv \left(w_0^{(n_r)}, w_1^{(n_r)}, \ldots, w_N^{(n_r)}\right)$ be the vector of equilibrium wages solving goods market clearing conditions

$$w_i L_i = \Lambda_{i0} w_0 n \widetilde{L}_0 + \sum_{j=1}^{N} \Lambda_{ij} w_j L_j, \quad i = 1,...,N, \tag{38}$$
$$w_0 \widetilde{L}_0 = \Lambda_{00} w_0 \widetilde{L}_0 + \sum_{j=1}^{N} \left(\Lambda_{0j}/n\right) w_j L_j. \tag{39}$$

for $n = n_r$.

Part 1: Wages do not explode. Suppose that the sequence of equilibrium wages of country 0, $\left\{w_0^{(n_r)}\right\}_{r=1}^{\infty}$, is unbounded. This implies that there exists a subsequence of equilibrium wages of country 0 that goes to infinity. Formally, there exists a subsequence of numbers $\{m_r\}_{r=1}^{\infty} \subseteq \{n_r\}_{r=1}^{\infty}$, with $\lim_{r\to\infty} m_r \to 0$, such that $\lim_{r\to\infty} w_0^{(m_r)} = \infty$.

Recall that we take the wage of country N as a numeraire and normalize $w_N^{(n_r)} = 1$ for all n_r. We can write

$$\frac{\Lambda_{0j}^{(m_r)}}{m_r} = \frac{a_{0j}}{m_r a_{0j} + \sum_{l=1}^{N-1} a_{lj} \left[w_0^{(m_r)}/w_l^{(m_r)}\right]^{\rho} + a_{Nj} \left[w_0^{(m_r)}\right]^{\rho}}, \quad j = 1, \ldots, N;$$

and

$$\Lambda_{00}^{(m_r)} = \frac{a_{00}}{a_{00} + \sum_{l=1}^{N-1} a_{l0} \left[w_0^{(m_r)} / w_l^{(m_r)} \right]^{\rho} + a_{N0} \left[w_0^{(m_r)} \right]^{\rho}}.$$

Since the last term in the denominators of $\Lambda_{0j}^{(m_r)} / m_r$ and $\Lambda_{00}^{(m_r)}$ goes to ∞ as $r \to \infty$ (given our assumption that $\rho > 0$), we get that $\lim_{r\to\infty} \left[\Lambda_{0j}^{(m_r)} / m_r \right] = 0$ for $j = 1, \ldots, N$, and $\lim_{r\to\infty} \Lambda_{00}^{(m_r)} = 0$.

Consider the case with $N = 1$. Since $w_0^{(m_r)}$ is the equilibrium wage of country 0, the trade balance condition of country 0,

$$\left(1 - \Lambda_{00}^{(m_r)}\right) w_0^{(m_r)} \widetilde{L}_0 = \left[\Lambda_{01}^{(m_r)} / m_r \right] L_1,$$

holds for each r. But as $r \to \infty$, we get that the left-hand side of the above expression goes to infinity, while the right-hand side goes to zero, leading to a contradiction. Thus, it cannot be the case that $\lim_{r\to\infty} w_0^{(m_r)} = \infty$, and so the sequence $\left\{ w_0^{(n_r)} \right\}_{r=1}^{\infty}$ is bounded.

Now consider the case with $N > 1$. In addition to supposing that $\lim_{r\to\infty} w_0^{(m_r)} = \infty$, suppose that for some $k \in \{1, \ldots, N-1\}$ sequence $\left\{ \left[\Lambda_{0k}^{(m_r)} / m_r \right] w_k^{(m_r)} \right\}_{r=1}^{\infty}$ is unbounded. Let $\{m'_r\}_{r=1}^{\infty} \subseteq \{m_r\}_{r=1}^{\infty}$ be a sequence such that $\lim_{r\to\infty} m'_r = 0$ and

$$\lim_{r\to\infty} \left[\Lambda_{0k}^{(m'_r)} / m'_r \right] w_k^{(m'_r)} = \infty. \tag{40}$$

Since as we have argued above $\lim_{r\to\infty} \left[\Lambda_{0k}^{(m_r)} / m_r \right] = 0$, we also have $\lim_{r\to\infty} \left[\Lambda_{0k}^{(m'_r)} / m'_r \right] = 0$. Then (40) implies that $\lim_{r\to\infty} w_k^{(m'_r)} = \infty$. Furthermore, we can write

$$\left[\Lambda_{0k}^{(m'_r)} / m'_r \right] w_k^{(m'_r)} = \frac{a_{0k}}{m'_r a_{0k} \left[w_k^{(m'_r)} \right]^{-1} + \sum_{l=1}^{N} a_{lk} \left[w_0^{(m'_r)} / w_l^{(m'_r)} \right]^{\rho} \left[w_k^{(m'_r)} \right]^{-1}}.$$

This allows us to see that the only way we can have (40) is to have

$$\lim_{r\to\infty} \left[w_0^{(m'_r)} / w_l^{(m'_r)} \right]^{\rho} \left[w_k^{(m'_r)} \right]^{-1} = 0 \text{ for } l = 1, \ldots, N. \tag{41}$$

Since $\left\{ w_0^{(m'_r)} \right\}_{r=1}^{\infty}$ is a subsequence of $\left\{ w_0^{(m_r)} \right\}_{r=1}^{\infty}$ and $\lim_{r\to\infty} w_0^{(m_r)} = \infty$, we also have

$\lim_{r\to\infty} w_0^{(m'_r)} = \infty$. Therefore, $\lim_{r\to\infty} \left[w_0^{(m'_r)}\right]^{\rho} = \infty$ and so (41) impies that

$$\lim_{r\to\infty} \left[w_l^{(m'_r)}\right]^{-\rho} \left[w_k^{(m'_r)}\right]^{-1} = 0 \text{ for } l = 1, \ldots, N.$$

This in turn implies that

$$\lim_{r\to\infty} \Lambda_{Nk}^{(m'_r)} w_k^{(m'_r)} = \lim_{r\to\infty} \frac{a_{Nk}}{m'_r a_{0k} \left[w_0^{(m'_r)}\right]^{-\rho} \left[w_k^{(m'_r)}\right]^{-1} + \sum_{l=1}^{N} a_{lk} \left[w_l^{(m'_r)}\right]^{-\rho} \left[w_k^{(m'_r)}\right]^{-1}} = \infty.$$

Consider then the trade balance condition of country N,

$$L_N = \Lambda_{N0}^{(m'_r)} w_0 m'_r \widetilde{L}_0 + \sum_{j=1}^{N} \Lambda_{Nj}^{(m'_r)} w_j^{(m'_r)} L_j.$$

The left-hand side of this condition is a finite number that does not depend on r. At the same time, given that $\lim_{r\to\infty} \Lambda_{Nk}^{(m'_r)} w_k^{(m'_r)} = \infty$, the right-hand side of this condition goes to ∞ as $r \to \infty$. We get a contradiction to the supposition that $\left\{ \left[\Lambda_{0k}^{(m_r)}/m_r\right] w_k^{(m_r)} \right\}_{r=1}^{\infty}$ is an unbounded sequence. Thus, $\left\{ \left[\Lambda_{0k}^{(m_r)}/m_r\right] w_k^{(m_r)} \right\}_{r=1}^{\infty}$ is bounded for all $k = 1, \ldots, N-1$.

Finally, consider the trade balance condition of country 0,

$$\left(1 - \Lambda_{00}^{(m_r)}\right) w_0^{(m_r)} \widetilde{L}_0 = \sum_{j=1}^{N} \left[\Lambda_{0j}^{(m_r)}/m_r\right] w_j^{(m_r)} L_j.$$

The left-hand side of this condition goes to ∞ as $r \to \infty$, while the right-hand side is bounded, which leads to a contradiction. Therefore, $\left\{ w_0^{(n_r)} \right\}_{r=1}^{\infty}$ is necessarily bounded.

Using a similar argument, we can show that, for any $i = 1, \ldots, N-1$, the sequence of equilibrium wages of country i, $\left\{ w_i^{(n_r)} \right\}_{r=1}^{\infty}$, is also bounded.

Part 2: Wages do not go to zero. Suppose that there exists a subsequence of equilibrium wages of country 0 that goes to zero. Formally, suppose that there exists a subsequence of numbers $\{m_r\}_{r=1}^{\infty} \subseteq \{n_r\}_{r=1}^{\infty}$, with $\lim_{r\to\infty} m_r = 0$, such that $\lim_{r\to\infty} w_0^{(m_r)} = 0$. The trade balance condition for country 0,

$$\left(1 - \Lambda_{00}^{(m_r)}\right) w_0^{(m_r)} \widetilde{L}_0 = \sum_{j=1}^{N} \left[\Lambda_{0j}^{(m_r)}/m_r\right] w_j^{(m_r)} L_j,$$

then implies that $\lim_{r\to\infty}\left[\Lambda_{0j}^{(m_r)}/m_r\right]w_j^{(m_r)}=0$ for all $j\in\{1,\ldots,N\}$, so that both the left-hand side and the right-hand side of this condition converge to zero. But then, since $w_N^{(m_r)}=1$ for all r, we get that $\lim_{r\to\infty}\left[\Lambda_{0N}^{(m_r)}/m_r\right]=0$.

Next, we can write $\Lambda_{0N}^{(m_r)}/m_r$ as

$$\frac{\Lambda_{0N}^{(m_r)}}{m_r}=\frac{a_{0N}}{m_r a_{0N}+\sum_{l=1}^{N-1}a_{lN}\left[w_0^{(m_r)}/w_l^{(m_r)}\right]^{\rho}+a_{NN}\left[w_0^{(m_r)}\right]^{\rho}}. \tag{42}$$

The first and the last term in the denominator of the above expression go to zero as $r\to\infty$. Then, if $N=1$, we get that $\lim_{r\to\infty}\left[\Lambda_{0N}^{(m_r)}/m_r\right]=\infty$, a contradiction. Thus, it cannot happen that $\lim_{r\to\infty}w_0^{(m_r)}=0$ in the case with $N=1$.

Consider the case with $N>1$. Given that $\lim_{r\to\infty}\left[\Lambda_{0N}^{(m_r)}/m_r\right]=0$, it has to be the case that at least one of the terms in the denominator of expression (42) converges to infinity. Hence, $\lim_{r\to\infty}\left[w_0^{(m_r)}/w_l^{(m_r)}\right]^{\rho}=\infty$ for at least one $l\in\{1,\ldots,N-1\}$.

We can write $\Lambda_{Nj}^{(m_r)}$ for any $j=1,\ldots,N$ as

$$\Lambda_{Nj}^{(m_r)}=\frac{a_{Nj}\left[w_0^{(m_r)}\right]^{\rho}}{m_r a_{0j}+\sum_{l=1}^{N}a_{lj}\left[w_0^{(m_r)}/w_l^{(m_r)}\right]^{\rho}}.$$

The numerator of the above expression converges to 0 as $r\to\infty$, while the denominator convergres to ∞, given that $\lim_{r\to\infty}\left[w_0^{(m_r)}/w_l^{(m_r)}\right]^{\rho}=\infty$ for at least one $l\in\{1,\ldots,N-1\}$. Thus, $\lim_{r\to\infty}\Lambda_{Nj}^{(m_r)}=0$ for all $j=1,\ldots,N$.

Finally, consider the trade balance condition of country N,

$$L_N=\Lambda_{N0}^{(m_r)}w_0^{(m_r)}m_r\widetilde{L}_0+\sum_{j=1}^{N}\Lambda_{Nj}^{(m_r)}w_j^{(m_r)}L_j. \tag{43}$$

The first term on the right-hand side of (43) goes to 0 as $r\to\infty$ because $\Lambda_{N0}^{(m_r)}\in[0,1]$ for any r, and $\lim_{r\to\infty}w_0^{(m_r)}=0$ and $\lim_{r\to\infty}m_r=0$. Also, each term $\Lambda_{Nj}^{(m_r)}w_j^{(m_r)}L_j$ on the right-hand side of (43) also goes to 0 as $r\to\infty$ because $\lim_{r\to\infty}\Lambda_{Nj}^{(m_r)}=0$ and, as we have shown in Part 1, sequence $\left\{w_j^{(m_r)}\right\}_{r=1}^{\infty}$ is bounded for each $j=1,\ldots,N$. Thus, the whole right-hand side of (43) goes to 0 as $r\to\infty$. At the same time, the left-hand side of (43) is a finite positive number that does not depend on r, which leads to a contradiction. Thus, the sequence of equilibrium wages of country 0, $\left\{w_0^{(n_r)}\right\}_{r=1}^{\infty}$, cannot have any subsequence converging to 0.

Using a similar argument, we can show that, for any $i = 1, \ldots, N-1$, the sequence of equilibrium wages of country i, $\left\{w_i^{(n_r)}\right\}_{r=1}^{\infty}$, cannot have any subsequence converging to 0.

Part 3. Convergence. Part 1 implies that the sequence of equilibrium wages $\left\{\boldsymbol{w}^{(n_r)}\right\}_{r=1}^{\infty}$ is bounded. Therefore, it has at least one converging subsequence.

Take any converging subsequence $\left\{\boldsymbol{w}^{(m_r)}\right\}_{r=1}^{\infty}$ of the sequence of equilibrium wages, where $\{m_r\}_{r=1}^{\infty} \subseteq \{n_r\}_{r=1}^{\infty}$ and $\lim_{r\to\infty} m_r = 0$. Let

$$\boldsymbol{w}^{(0)} = \left(w_0^{(0)}, w_1^{(0)}, \ldots, w_N^{(0)}\right) \equiv \lim_{r\to\infty} \boldsymbol{w}^{(m_r)}.$$

Part 2 implies that $w_i^{(0)} > 0$ for all $i = 0, 1, \ldots, N$. This, in turn, implies that we have well-defined limits

$$\Lambda_{ij}^{(0)} \equiv \lim_{r\to\infty} \Lambda_{ij}^{(m_r)} = \frac{a_{ij}\left[w_i^{(0)}\right]^{-\rho}}{\sum_{l=1}^{N} a_{lj}\left[w_l^{(0)}\right]^{-\rho}} \in (0,1), \quad \text{for } i,j = 1, ..., N; \tag{44}$$

$$\Lambda_{i0}^{(0)} \equiv \lim_{r\to\infty} \Lambda_{i0}^{(m_r)} = \frac{a_{i0}\left[w_i^{(0)}\right]^{-\rho}}{a_{00}\left[w_0^{(0)}\right]^{-\rho} + \sum_{l=1}^{N} a_{l0}\left[w_l^{(0)}\right]^{-\rho}} \in (0,1), \quad \text{for } i = 1, ..., N; \tag{45}$$

$$\Lambda_{00}^{(0)} \equiv \lim_{r\to\infty} \Lambda_{00}^{(m_r)} = \frac{a_{00}\left[w_0^{(0)}\right]^{-\rho}}{a_{00}\left[w_0^{(0)}\right]^{-\rho} + \sum_{l=1}^{N} a_{l0}\left[w_l^{(0)}\right]^{-\rho}} \in (0,1); \tag{46}$$

as well as

$$\tilde{\Lambda}_{0j}^{(0)} \equiv \lim_{r\to\infty} \frac{\Lambda_{0j}^{(m_r)}}{m_r} = \frac{a_{0j}\left[w_0^{(0)}\right]^{-\rho}}{\sum_{l=1}^{N} a_{lj}\left[w_l^{(0)}\right]^{-\rho}} \in (0,\infty), \quad \text{for } j = 1, ..., N. \tag{47}$$

Substituting the vector of equilibrium wages $\boldsymbol{w}^{(m_r)}$ into the equilibrium equations (38) for countries $i = 1, \ldots, N$, and taking the limit $r \to \infty$, we get equalities

$$w_i^{(0)} L_i = \sum_{j=1}^{N} \Lambda_{ij}^{(0)} w_j^{(0)} L_j, \quad i = 1, ..., N, \tag{48}$$

where both the left-hand side and the right-hand side are finite positive numbers. Given the expression (44) for $\Lambda_{ij}^{(0)}$, the equalities (48) can also be viewed as a system of equations in wages, $(w_1, \ldots, w_N)$,

$$w_i L_i = \sum_{j=1}^{N} \frac{a_{ij} w_i^{-\rho}}{\sum_{l=1}^{N} a_{lj} w_l^{-\rho}} w_j L_j, \quad i = 1, ..., N,$$

that is solved for $(w_1, \ldots, w_N) = \left(w_1^{(0)}, \ldots, w_N^{(0)}\right)$. This system of equations, is of course, the equilibrium system of equations for our gravity model that consists of countries $1, \ldots, N$ only and does not include country 0. We know from Proposition 1 from the main text that this system of equations has a unique solution (given our normalization $w_N = 1$).

Next, substituting the vector of equilibrium wages $\boldsymbol{w}^{(m_r)}$ into the equilibrium equilibrium equation (39) for country 0 and taking the limit $r \to \infty$, we get equality

$$w_0^{(0)} \widetilde{L}_0 = \Lambda_{00}^{(0)} w_0^{(0)} \widetilde{L}_0 + \sum_{j=1}^{N} \widetilde{\Lambda}_{0j}^{(0)} w_j^{(0)} L_j,$$

where again both the left-hand side and the right-hand side are finite positive numbers. Given the expressions (46) and (47) for $\Lambda_{00}^{(0)}$ and $\widetilde{\Lambda}_{0j}^{(0)}$ and for a fixed vector $\left(w_1^{(0)}, \ldots, w_N^{(0)}\right)$ of wages of countries $1, \ldots, N$, we can view this equality as an equation in wage of country 0, w_0,

$$w_0 \widetilde{L}_0 = \frac{a_{00} w_0^{-\rho}}{a_{00} w_0^{-\rho} + \sum_{l=1}^{N} a_{l0} \left[w_l^{(0)}\right]^{-\rho}} w_0 \widetilde{L}_0 + \left\{ \sum_{j=1}^{N} \frac{a_{0j} w_j^{(0)} L_j}{\sum_{l=1}^{N} a_{lj} \left[w_l^{(0)}\right]^{-\rho}} \right\} w_0^{-\rho},$$

that is solved for $w_0 = w_0^{(0)}$. It is easy to see that this equation in w_0 has at most one solution. Indeed, we can rearrange it to get

$$\frac{\sum_{l=1}^{N} a_{l0} w_l^{(0)}}{a_{00} w_0^{-\rho} + \sum_{l=1}^{N} a_{l0} w_l^{(0)}} w_0 \widetilde{L}_0 = \left\{ \sum_{j=1}^{N} \frac{a_{0j}}{\sum_{l=1}^{N} a_{lj} \left[w_l^{(0)}\right]^{-\rho}} w_j^{(0)} L_j \right\} w_0^{-\rho}. \tag{49}$$

The left-hand side of (49) is an increasing function of w_0, while the right-hand side is a decreasing function of w_0. Thus, there can be only one w_0 that equates the left-hand side and the right-hand side of (49).

Collecting our results, we get that any converging subsequence of equilibrium wages converges to the same limit. This implies that the sequence of equilibrium wages itself converges to this limit. The wage vector $\left(w_1^{(0)}, \ldots, w_N^{(0)}\right)$ of countries $1, \ldots, N$ obtained in the limit constitutes the equilibrium of the economy with countries $1, \ldots, N$ only. Given this wage vector, the wage vector $w_0^{(0)}$ of country 0 obtained in the limit constitutes the equilibrium of the SOE.

5 Comparative Statics in the Multi-Sector SOE without Policy Instruments

Existence and Uniqueness. Equilibrium existence and uniqueness in the SOE model for $\alpha_k \in [0,1]$ without taxes and subsidies has been established by Kucheryavyy et al. (2023, Section 4.7 in the Online Appendix). Here we repeat the proof for $\alpha_k \in [0,1)$ for completeness, and to use this proof as a starting point for the comparative statics analysis.

Without policy instruments, the SOE's goods market clearing condition becomes

$$wL_k = \lambda_k \beta_k wL + E_k, \tag{50}$$

where $E_k \equiv A_k^{\varepsilon_k} w^{-\zeta_k} L_k^{\alpha_k} D_k$ denotes SOE's exports in industry k,

$$\lambda_k = \frac{A_k^{\varepsilon_k} w^{-\zeta_k} L_k^{\alpha_k}}{A_k^{\varepsilon_k} w^{-\zeta_k} L_k^{\alpha_k} + \mathcal{P}_k^{-\zeta_k}} \tag{51}$$

is the SOE's expenditure share on domestic goods in sector k, and where $D_k \equiv \sum_{j=1}^{N} D_{j,k}$ and $\mathcal{P}_k \equiv \left[\sum_{l=1}^{N} p_{l,k}^{-\zeta_k}\right]^{-1/\zeta_k}$. Dividing both sides of the goods market clearing condition by $wL_k^{\alpha_k}$, we get

$$L_k^{1-\alpha_k} = \frac{A_k^{\varepsilon_k} w^{-\zeta_k}}{A_k^{\varepsilon_k} w^{-\zeta_k} L_k^{\alpha_k} + \mathcal{P}_k^{-\zeta_k}} \beta_k L + A_k^{\varepsilon_k} w^{-\zeta_k - 1} D_k. \tag{52}$$

For fixed w, D_k, and $\mathcal{P}_k$, and for $\alpha_k \in [0,1)$, the left-hand side of (52) increases with L_k, while the right-hand side decreases with L_k if $\alpha_k \in (0,1)$, and is a constant if $\alpha_k = 0$. Furthermore, as L_k ranges from 0 to ∞, the left-hand side of (52) also ranges from 0 to ∞, while the right-hand side is bounded from above and away from zero. Therefore, there exists a unique positive finite L_k that solves (52). Thus L_k can be regarded as a function of w, D_k and $\mathcal{P}_k$, implicitly defined by equation (52). We denote this function by $L_k(w; D_k, \mathcal{P}_k)$ but drop the dependence on w, D_k and $\mathcal{P}_k$ in most expressions for brevity.

Observe that for a fixed L_k, as w goes to 0, the right-hand side of (52) goes to ∞, and as w goes to ∞, the right-hand side of (52) goes to 0. Therefore, as w ranges from 0 to ∞, $L_k(w; D_k, \mathcal{P}_k)$ ranges from ∞ to 0.

Differentiating (52) with respect to $\ln w$, $\ln D_k$ and $\ln \mathcal{P}_k$, yields

$$\left[(1-\alpha_k)\, wL_k + \alpha_k \lambda_k^2 \beta_k wL\right] \frac{\partial \ln L_k}{\partial \ln w} = -\zeta_k (1-\lambda_k)\, \lambda_k \beta_k wL - (1+\zeta_k)\, E_k,$$
$$\left[(1-\alpha_k)\, wL_k + \alpha_k \lambda_k^2 \beta_k wL\right] \frac{\partial \ln L_k}{\partial \ln D_k} = E_k,$$
$$\left[(1-\alpha_k)\, wL_k + \alpha_k \lambda_k^2 \beta_k wL\right] \frac{\partial \ln L_k}{\partial \ln \mathcal{P}_k} = \zeta_k \lambda_k^2 \beta_k wL.$$

These expressions imply that $L_k(w; D_k, \mathcal{P}_k)$ decreases in w and increases in D_k and $\mathcal{P}_k$.

Next, we can write the SOE's labor market clearing condition as

$$\sum_k L_k(w; D_k, \mathcal{P}_k) = L \tag{53}$$

Given that the left-hand side of (53) decreases in w and falls from ∞ to 0 as w increases from 0 to ∞, there is a unique w that solves (53).

Effects on Labor Allocations and Wages. We have for any k and r,

$$\frac{d \ln L_k}{d \ln D_r} = \delta_{rk} \frac{\partial \ln L_k}{\partial \ln D_k} + \frac{\partial \ln L_k}{\partial \ln w} \cdot \frac{d \ln w}{d \ln D_r}, \tag{54}$$
$$\frac{d \ln L_k}{d \ln \mathcal{P}_r} = \delta_{rk} \frac{\partial \ln L_k}{\partial \ln \mathcal{P}_k} + \frac{\partial \ln L_k}{\partial \ln w} \cdot \frac{d \ln w}{d \ln \mathcal{P}_r}, \tag{55}$$

where δ_{rk} is 1 if $r = k$, and 0 otherwise. Differentiating (53) with respect to D_r and $\mathcal{P}_r$ for some $r = 1, \ldots, K$, and using (54) and (55), yields

$$\left(\sum_{k=1}^{K} L_k \frac{\partial \ln L_k}{\partial \ln w}\right) \cdot \frac{d \ln w}{d \ln D_r} = -L_r \frac{\partial \ln L_r}{\partial \ln D_r},$$
$$\left(\sum_{k=1}^{K} L_k \frac{\partial \ln L_k}{\partial \ln w}\right) \cdot \frac{d \ln w}{d \ln \mathcal{P}_r} = -L_r \frac{\partial \ln L_r}{\partial \ln \mathcal{P}_r}.$$

These expressions imply that $d \ln w / d \ln D_r > 0$ and $d \ln w / d \ln \mathcal{P}_r > 0$. Then, going back to expressions (54) and (55), we get $d \ln L_k / d \ln D_r < 0$ and $d \ln L_k / d \ln \mathcal{P}_r < 0$ for all $k \neq r$. In turn, these results combined with the labor market clearing condition imply that $d \ln L_k / d \ln D_k > 0$ and $d \ln L_k / d \ln \mathcal{P}_k > 0$ for any k.

Effects on Trade Flows. Differentiating trade shares λ_k, yields

$$\frac{1}{1-\lambda_k} \cdot \frac{d \ln \lambda_k}{d \ln D_r} = -\zeta_k \frac{d \ln w}{d \ln D_r} + \alpha_k \frac{d \ln L_k}{d \ln D_r}, \tag{56}$$

$$\frac{1}{1-\lambda_k} \cdot \frac{d \ln \lambda_k}{d \ln \mathcal{P}_r} = \delta_{rk}\zeta_k - \zeta_k \frac{d \ln w}{d \ln \mathcal{P}_r} + \alpha_k \frac{d \ln L_k}{d \ln \mathcal{P}_r}; \tag{57}$$

and differentiating exports E_k, yields

$$\frac{d \ln E_k}{d \ln D_r} = \delta_{rk} D_r + \frac{1}{1-\lambda_k} \cdot \frac{d \ln \lambda_k}{d \ln D_r},$$

$$\frac{d \ln E_k}{d \ln \mathcal{P}_r} = -\delta_{rk}\zeta_k + \frac{1}{1-\lambda_k} \cdot \frac{d \ln \lambda_k}{d \ln \mathcal{P}_r}.$$

Consider $d \ln \lambda_k / d \ln D_r$ and $d \ln E_k / d \ln D_r$ first. Since, $d \ln w / d \ln D_r > 0$ for all r and $d \ln L_k / d \ln D_r < 0$ for $r \neq k$, we get $d \ln \lambda_k / d \ln D_r < 0$ for $r \neq k$. From here, we also immediately get that $d \ln E_k / d \ln D_r < 0$ for $r \neq k$. At the same time, the effect of an increase in D_k on λ_k is ambiguous.

If $d \ln \lambda_k / d \ln D_k > 0$, then $d \ln E_k / d \ln D_k > 0$. Suppose that $d \ln \lambda_k / d \ln D_k < 0$. Adding up goods market clearing conditions (50) across k, using the labor market clearing condition $\sum_k L_k = L$, and rearranging terms, gives the SOE's trade balance condition,

$$(1-\lambda_k)\,\beta_k w L + \sum_{r \neq k} (1-\lambda_r)\,\beta_r w L = E_k + \sum_{r \neq k} E_r. \tag{58}$$

The first term on the left-hand side of (58) is increasing in D_k because w is increasing in D_k and $1-\lambda_k$ is increasing in D_k by our supposition. The second term on the left-hand side of (58) is also increasing in D_k. Thus, the whole left-hand side of D_k is increasing in D_k. At the same time, the second term on the right-hand side of (58) is decreasing in D_k. Therefore, it must be the case that the first term on the right-hand side of (58) is increasing in D_k. Thus, we get that E_k is increasing in D_k regardless of whether λ_k is increasing or decreasing in D_k.

Now consider $d \ln \lambda_k / d \ln \mathcal{P}_r$ and $d \ln E_k / d \ln \mathcal{P}_r$. As with $d \ln \lambda_k / d \ln D_r$ and $d \ln E_k / d \ln D_r$, it is easy to see that $d \ln \lambda_k / d \ln \mathcal{P}_r < 0$ and $d \ln E_k / d \ln \mathcal{P}_r < 0$ for $r \neq k$. Suppose that $d \ln \lambda_k / d \ln \mathcal{P}_k < 0$. Then $d \ln E_k / d \ln \mathcal{P}_k < 0$. But these two effects then imply that the left-hand side of the trade balance condition (58) is increasing in $\mathcal{P}_k$, while the right-hand side is decreasing in $\mathcal{P}_k$, a contradiction. Thus, it has to be the case that $d \ln \lambda_k / d \ln \mathcal{P}_k > 0$. At the same time, the effect of an increase in $\mathcal{P}_k$ on E_k is ambiguous.

Effects on Welfare. Without the policy instruments, the SOE's price index in sector k becomes

$$P_k = \delta_k \cdot (\beta_k w L)^{1-\frac{\zeta_k}{\varepsilon_k}} \left[A_k^{\varepsilon_k} w^{-\zeta_k} L_k^{\alpha_k} + \mathcal{P}_k^{-\zeta_k} \right]^{-\frac{1}{\varepsilon_k}}.$$

Using expression (51) for λ_k, this can be written as

$$P_k = \delta_k \cdot (\beta_k L)^{1-\zeta_k/\varepsilon_k} w A_k^{-1} L_k^{-\alpha_k/\varepsilon_k} \lambda_k^{1/\varepsilon_k},$$

and so the real wage is given by

$$W \equiv \frac{w}{P} = \kappa_W \prod_{k=1}^{K} L_k^{\alpha_k \beta_k/\varepsilon_k} \lambda_k^{-\beta_k/\varepsilon_k},$$

where

$$\kappa_W \equiv \prod_{k=1}^{K} \delta_k^{-\beta_k} \beta_k^{\zeta_k \beta_k / \overline{\varepsilon_k}} \left(\frac{L}{f_k} \right)^{-\left(1-\frac{\zeta_k}{\varepsilon_k}\right)\beta_k} A_k^{\beta_k}$$

is a composite parameter. Letting W^A denote welfare in the counterfactual corresponding to autarky and letting $\text{GT} \equiv W/W^A$ denote the gains from trade, we then have

$$\text{GT} = \prod_{k=1}^{K} \left(\frac{L_k}{\beta_k L} \right)^{\alpha_k \beta_k/\varepsilon_k} \lambda_k^{-\beta_k/\varepsilon_k},$$

where we used the fact that in autarky SOE allocates $\beta_k L$ units of labor to industry k. Therefore,

$$\frac{d \ln \text{GT}}{d \ln D_r} = \sum_{k=1}^{K} \frac{\alpha_k \beta_k}{\varepsilon_k} \cdot \frac{d \ln L_k}{d \ln D_r} - \sum_{k=1}^{K} \frac{\beta_k}{\varepsilon_k} \cdot \frac{d \ln \lambda_k}{d \ln D_r},$$
$$\frac{d \ln \text{GT}}{d \ln \mathcal{P}_r} = \sum_{k=1}^{K} \frac{\alpha_k \beta_k}{\varepsilon_k} \cdot \frac{d \ln L_k}{d \ln \mathcal{P}_r} - \sum_{k=1}^{K} \frac{\beta_k}{\varepsilon_k} \cdot \frac{d \ln \lambda_k}{d \ln \mathcal{P}_r}.$$

6 Optimal Policy in the Multi-Sector SOE

To simplify the algebra, we assume that the SOE imposes uniform tariffs on imports from all sources, $\bar{t}_{i,k}^m = \bar{t}_k^m$ for all i, and levies the same export tax on goods destined for all markets, $\bar{t}_{j,k}^x = \bar{t}_k^x$ for all j. Consequently, the consumer price index for imports in industry k becomes $\left(\bar{t}_k^m\right)^{-\zeta_k} p_k^{-\zeta_k}$, and the export demand in industry k collapses into the term $\left(\bar{s}_k/\bar{t}_k^x\right)^{-(\zeta_k+1)} w^{-\rho_k} L_k^{\alpha_k} A_k^{\varepsilon_k} D_k$, where $\mathcal{P}_k^{-\zeta_k} = \sum_{i=1}^{N} p_{i,k}^{-\zeta_k}$ and $D_k = \sum_{j=1}^{N} D_{j,k}$.

We can write the SOE tax revenue as

$$T = \sum_{k=1}^{K} \left(T_k^m + T_k^x + T_k^L \right), \tag{59}$$

where

$$\begin{aligned} T_k^m &\equiv \frac{\bar{t}_k^m - 1}{\bar{t}_k^m} \left(1 - \lambda_k\right) \beta_k \left(wL + T\right), \\ T_k^x &\equiv \left(1 - \bar{t}_k^x\right) \left(\frac{\bar{s}_k}{\bar{t}_k^x}\right)^{-\zeta_k} w^{-\zeta_k} L_k^{\alpha_k} A_k^{\varepsilon_k} D_k, \\ T_k^L &\equiv \left(\bar{s}_k - 1\right) wL_k. \end{aligned}$$

Then the SOE's goods market clearing condition in sector k,

$$wL_k = \frac{\lambda_k}{\bar{s}_k} \beta_k \left(wL + T\right) + \left(\frac{\bar{s}_k}{\bar{t}_k^x}\right)^{-\zeta_k - 1} w^{-\zeta_k} L_k^{\alpha_k} A_k^{\varepsilon_k} D_k,$$

can be written in terms of T_k^m, T_k^x, and T_k^L as

$$\frac{\bar{s}_k}{\bar{s}_k - 1} T_k^L = \frac{\lambda_k}{1 - \lambda_k} \cdot \frac{\bar{t}_k^m}{\bar{t}_k^m - 1} T_k^m + \frac{\bar{t}_k^x}{1 - \bar{t}_k^x} T_k^x. \tag{60}$$

Using (59), (60), and the SOE's labor market-clearing condition $L = \sum_k L_k$ to rewrite the SOE's household budget constraint $X = wL + T$, the SOE system is summarized as

$$\begin{aligned} & W = X / \left(LP\right), \\ & X = wL + T = Y/Z, \\ & \quad Y \equiv \sum_k \left(Y_k^x + Y_k^L\right), Y_k^x \equiv \frac{\bar{t}_k^m - \bar{t}_k^x}{\bar{t}_k^m} \cdot \frac{1}{1 - \bar{t}_k^x} T_k^x, Y_k^L \equiv \frac{1}{\bar{t}_k^m} \cdot \frac{\bar{s}_k}{\bar{s}_k - 1} T_k^L, Z \equiv \sum_k \frac{\beta_k}{\bar{t}_k^m}, \\ & \quad T_k^x \equiv A_k^{\varepsilon_k} w^{-\zeta_k} L_k^{\alpha_k} D_k \left(1 - \bar{t}_k^x\right) \left(\frac{\bar{s}_k}{\bar{t}_k^x}\right)^{-\zeta_k}, T_k^L \equiv \left(\bar{s}_k - 1\right) wL_k, \\ & P = \prod_k \left(P_k / \beta_k\right)^{\beta_k}, \\ & \quad P_k = \delta_k \cdot \left(\beta_k X\right)^{1 - \zeta_k / \varepsilon_k} \left(\bar{s}_k^{-\zeta_k} A_k^{\varepsilon_k} w^{-\zeta_k} L_k^{\alpha_k} + \left[\bar{t}_k^m\right]^{-\zeta_k} \mathcal{P}_k^{-\zeta_k} \right)^{-1/\varepsilon_k}, \end{aligned}$$

$$\lambda_k = \frac{\bar{s}_k^{-\zeta_k} A_k^{\varepsilon_k} w^{-\zeta_k} L_k^{\alpha_k}}{\bar{s}_k^{-\zeta_k} A_k^{\varepsilon_k} w^{-\zeta_k} L_k^{\alpha_k} + \left[\bar{t}_k^m\right]^{-\zeta_k} \mathcal{P}_k^{-\zeta_k}},$$
$$L = \sum_k L_k.$$

Simply applying the local hat algebra $\widehat{x} \equiv d \ln x$ to the SOE system, the logarithmic change in the SOE's welfare is given by

$$\widehat{W} = \frac{\partial \ln W}{\partial \ln w}\widehat{w} + \sum_k \frac{\partial \ln W}{\partial \ln L_k}\widehat{L}_k + \sum_k \frac{\partial \ln W}{\partial \ln \bar{t}_k^m}\widehat{\bar{t}}_k^m + \sum_k \frac{\partial \ln W}{\partial \ln \bar{t}_k^x}\widehat{\bar{t}}_k^x + \sum_k \frac{\partial \ln W}{\partial \ln \bar{s}_k}\widehat{\bar{s}}_k;$$
$$\frac{\partial \ln W}{\partial \ln w} \equiv \sum_k \left[\frac{\sum_h \beta_h \zeta_h / \varepsilon_h}{X \sum_h \beta_h / \bar{t}_h^m}\left(-Y_k^x \zeta_k + Y_k^L\right) - \frac{\beta_k}{\varepsilon_k}\lambda_k \zeta_k\right],$$
$$\frac{\partial \ln W}{\partial \ln L_k} \equiv \frac{\sum_h \beta_h \zeta_h / \varepsilon_h}{X \sum_h \beta_h / \bar{t}_h^m}\left(Y_k^x \alpha_k + Y_k^L\right) + \frac{\beta_k}{\varepsilon_k}\lambda_k \alpha_k,$$
$$\frac{\partial \ln W}{\partial \ln \bar{t}_k^m} \equiv \frac{\sum_h \beta_h \zeta_h / \varepsilon_h}{X \sum_h \beta_h / \bar{t}_h^m}\left(Y_k^x \frac{\bar{t}_k^x}{\bar{t}_k^m - \bar{t}_k^x} - Y_k^L + X\frac{\beta_k}{\bar{t}_k^m}\right) - \frac{\beta_k}{\varepsilon_k}\left(1-\lambda_k\right)\zeta_k,$$
$$\frac{\partial \ln W}{\partial \ln \bar{t}_k^x} \equiv \frac{\sum_h \beta_h \zeta_h / \varepsilon_h}{X \sum_h \beta_h / \bar{t}_h^m} Y_k^x \left(\zeta_k - \frac{\bar{t}_k^x}{\bar{t}_k^m - \bar{t}_k^x}\right),$$
$$\frac{\partial \ln W}{\partial \ln \bar{s}_k} \equiv \frac{\sum_h \beta_h \zeta_h / \varepsilon_h}{X \sum_h \beta_h / \bar{t}_h^m}\left(-Y_k^x \zeta_k + Y_k^L\right) - \frac{\beta_k}{\varepsilon_k}\lambda_k \zeta_k.$$

This implies that the optimal policy should satisfy the following first-order conditions

$$0 = \frac{\partial \ln W}{\partial \ln w}\frac{\partial \ln w}{\partial \ln \bar{t}_k^m} + \sum_h \frac{\partial \ln W}{\partial \ln L_h}\frac{\partial \ln L_h}{\partial \ln \bar{t}_k^m} + \frac{\partial \ln W}{\partial \ln \bar{t}_k^m}, \tag{61}$$
$$0 = \frac{\partial \ln W}{\partial \ln w}\frac{\partial \ln w}{\partial \ln \bar{t}_k^x} + \sum_h \frac{\partial \ln W}{\partial \ln L_h}\frac{\partial \ln L_h}{\partial \ln \bar{t}_k^x} + \frac{\partial \ln W}{\partial \ln \bar{t}_k^x}, \tag{62}$$
$$0 = \frac{\partial \ln W}{\partial \ln w}\frac{\partial \ln w}{\partial \ln \bar{s}_k} + \sum_h \frac{\partial \ln W}{\partial \ln L_h}\frac{\partial \ln L_h}{\partial \ln \bar{s}_k} + \frac{\partial \ln W}{\partial \ln \bar{s}_k}. \tag{63}$$

Note that w and L_k are endogenously determined from the goods and labor market-clearing conditions.

We guess and verify that the optimal policy entails

$$\bar{t}_k^m = \bar{t}\varepsilon_k/\zeta_k \quad \Rightarrow \quad \frac{\sum_h \beta_h \zeta_h/\varepsilon_h}{X\sum_h \beta_h/\bar{t}_h^m} = \frac{\bar{t}}{X}$$

and

$$\zeta_k = \frac{\bar{t}_k^x}{\bar{t}_k^m - \bar{t}_k^x} \quad \Rightarrow \quad \bar{t}_k^x = \bar{t}\frac{\varepsilon_k}{1+\zeta_k} \quad \Rightarrow \quad \frac{\partial \ln W}{\partial \ln \bar{t}_k^x} = 0.$$

Then, at the proposed optimal policy, we can easily verify that $\frac{\partial \ln W}{\partial \ln \bar{t}_k^m} = \frac{\partial \ln W}{\partial \ln \bar{s}_k} = \frac{\partial \ln W}{\partial \ln w} = 0$, and

$$\sum_k \frac{\partial \ln W}{\partial \ln L_k}\widehat{L}_k = \frac{wL}{X}\sum_k \frac{\alpha_k + \zeta_k}{\varepsilon_k}\bar{s}_k \frac{L_k}{L}\widehat{L}_k.$$

If $\bar{s}_k = \bar{s}\varepsilon_k/\left(\alpha_k + \zeta_k\right)$, then the above expression becomes

$$\sum_k \frac{\partial \ln W}{\partial \ln L_k}\widehat{L}_k = \frac{wL\bar{s}}{X}\sum_k \frac{L_k}{L}\widehat{L}_k = 0,$$

where the last equality follows from the labor market-clearing condition $L = \sum_k L_k$. This establishes that (61), (62), and (63) are satisfied at the proposed optimal policy.

References

Kucheryavyy, Konstantin, Gary Lyn, and Andrés Rodríguez-Clare, "Grounded by Gravity: A Well-Behaved Trade Model with Industry-Level Economies of Scale," *American Economic Journal: Macroeconomics*, April 2023, *15* (2), 372–412.

Mas-Colell, Andreu, Michael D. Whinston, and Jerry R. Green, *Microeconomic Theory*, Oxford University Press, 1995.