Characterizing Optimality in Dynamic Settings: A Monotonicity-based Approach *

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Abstract

We develop a novel analytical method for studying optimal paths in dynamic optimization problems under general monotonicity conditions. The method centers on a *locator function*—constructed directly from the model's primitives—whose roots identify interior steady states and whose slope determines local stability. Under strict concavity of the payoff function, the locator function also characterizes basins of attraction, yielding a complete description of qualitative dynamics. Without concavity, it still delivers sharp results: if it is single-crossing from above, its root identifies a globally stable steady state; if it is inverted-U shaped with two interior roots, only the higher root can be a locally stable interior steady state. The locator function further enables comparative statics of steady states through direct analysis of its derivatives. These results are obtained without solving the full dynamic program. We illustrate the approach using a generalized neoclassical growth model and a learning-by-doing economy.

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1 Introduction

Dynamic optimization problems are central to many areas of economic analysis, from growth and consumption to investment and environmental policy. In many such settings, the goal is not to compute the full optimal path—that is, the sequence of state and control variables over time that solves the dynamic optimization problem—but rather to understand its qualitative features, such as the location and stability of steady states and the direction in which the system evolves from a given initial condition. Standard methods based on dynamic programming characterize optimal behavior via the Bellman equation, but solving this equation typically requires numerical methods and often yields limited insight into qualitative dynamics, especially in the presence of multiple steady states.

This paper's contribution is methodological. It introduces a new analytical tool—the *locator function*—that helps to characterize steady states and their stability directly from the model's primitives within a large class of economies satisfying a set of monotonicity conditions described below. The function is simple and tractable, depending only on the per-period payoff, π, and the discount factor, δ. Formally, it is defined as a scalar function of the state variable s constructed from these primitives,

$$\mathcal{L}(s, \delta) = \pi_2(s, s) + \delta \pi_1(s, s),$$

where π_l denotes the partial derivative with respect to the l-th argument. Focusing on this function provides a direct way to characterize key features of the optimal path without solving the full dynamic program. This is formalized in our main theorem, which shows that the set of roots of the locator function contains the set of interior steady states, and that local stability is determined by the sign of the derivative at those roots — a negative derivative indicates stability, whereas a positive derivative implies instability.

While this result establishes the basic link between the locator function and stability, sharper insights can be obtained under additional assumptions. A natural starting point is the standard textbook case of strict concavity in the payoff function (e.g., Stokey et al. (1989)). In this setting, we show that—with a mild additional condition—the locator function not only identifies the interior steady states but also fully characterizes their basins of attraction. That is, we obtain a complete description of the qualitative dynamics: for any initial condition, we can determine the direction of motion and the limiting behavior of the optimal path. This goes beyond the standard linearization approach, which approximates the system around a steady state and assesses stability through the eigenvalues of the Jacobian. Linearization establishes only the existence of basins of attraction but does not describe them. By contrast, our method provides a complete and tractable characterization.

Unlike the textbook linearization method (cf. Stokey et al. (1989)), our approach is not confined to the strictly concave case. This considerably broadens the scope of analysis, as many economically relevant models feature non-concave per-period payoffs—for instance, growth models with non-concave production functions (e.g., Skiba (1978) and Azariadis and Drazen (1990)), a possibility supported by empirical evidence (Durlauf and Johnson, 1995). In such environments, our method can still deliver tractable results on the existence and stability of steady states. First, the locator function provides easy-to-verify sufficient conditions for the existence and location of a globally stable steady state. In particular, if the function is single-crossing from above—i.e., it has a unique root and is positive to the left and negative to the right—then any optimal path converges globally to that root from any interior initial condition. This allows applied researchers to verify global stability directly from the payoff function and its partial derivatives, without solving the full dynamic program. Second, we consider what is arguably the typical non-convex case: an inverted-U shaped locator function with two interior roots. In this setting, if an interior locally stable steady state exists, it must correspond to the highest root; no other interior root can be locally stable.

Finally, we develop a tractable method for comparative statics based on the locator function. Specifically, the effect of parameter changes on stable steady states can be derived directly from the derivatives of the locator function. This greatly simplifies the analysis: rather than solving the full dynamic program or characterizing the policy correspondence, one can determine how steady states shift by inspecting the sign of a single derivative. In contrast, standard approaches often rely on numerical solutions or indirect arguments.

Our approach applies to economies that satisfy two key monotonicity conditions on the payoff function. First, π must be monotonic in the current state. Second, current and future states must be complementary, in the sense that the marginal payoff from a higher future state increases with the level of the current state—that is, $\pi_2(s, s')$ is increasing in s. These monotonicity conditions offer an alternative to the concavity assumptions common in textbook analysis, yielding tractable tools for a broader class of economies.

To illustrate our approach, we present two examples that highlight its ability to characterize qualitative dynamics in environments with non-convexities. The first revisits the neoclassical growth model with potentially non-convex production, as in Skiba (1978) and Azariadis and Drazen (1990), and allows for kinks. In the smooth convex–concave case, Dechert and Nishimura (2012) fully characterize the optimal path using advanced tools; our approach recovers these results more directly and extends them to settings with kinks and more general non-concavities.

The second example connects to the literature on learning-by-doing and inefficient specialization (e.g., Krugman (1987), Lucas Jr (1988), Young (1991)). From the planner's perspective (cf. Bardhan (1971), Melitz (2005)), we derive a simple condition on the learning elasticity that governs the shape of the locator function and the resulting dynamics. When the elasticity is low relative to standard curvature forces—decreasing marginal productivity and diminishing marginal utility (or revenue)—the locator function is single-crossing from above, implying a unique, globally stable interior steady state. When it is high, the locator function becomes inverted-U shaped and may admit multiple interior roots; only the highest can be locally stable, and the myopic-planner locator function provides a conservative estimate of the left boundary of its basin of attraction.

While these examples illustrate how complementarities shape dynamics in specific settings, the monotonicity condition $\pi_{12}(s, s') > 0$ is a pervasive feature of dynamic models across many areas of economics. In the consumption–savings problem, concavity of utility and a linear intertemporal budget constraint imply that higher wealth lowers the marginal utility cost of increasing future assets, generating intertemporal complementarity even without technological non-convexities (e.g., Bewley (1986), Huggett (1993), Aiyagari (1994)). Similar forces arise in models with financial frictions, where borrowing or investment costs decline with wealth or capital, so that higher net worth raises the marginal payoff from future accumulation even with linear preferences (e.g., Kiyotaki and Moore (1997), Grüne et al. (2005)). They also arise in models of human capital accumulation or adjustment costs, in which higher current stocks reduce the marginal cost of further accumulation (e.g., Ben-Porath (1967), Hayashi (1982)). Finally, they are central in renewable resource models, where preservation is more productive at higher stock levels; see Clark (1973). These examples underscore that the monotonicity conditions underlying our analysis are broadly applicable.

Relation to the literature. We are not the first to study the qualitative behavior of solutions to dynamic optimization problems. We begin by reviewing existing approaches in the literature and contrasting them with our method.

One approach relies on techniques tailored to the specific environment under study. This strategy is commonly used, for example, to characterize optimal paths in neoclassical growth models. Within this line of work, the papers most closely related to ours are Skiba (1978) and Dechert and Nishimura (2012). As we show in detail below, even though our method is not designed specifically for this setting, it nonetheless delivers results comparable to those in Dechert and Nishimura (2012). Although this is only one example, it illustrates that our approach can match the precision of environment-specific methods while maintaining greater generality.

Among general-purpose tools, two main approaches have been used to study the dynamics induced by the optimal policy correspondence—one based on Lyapunov functions and the other on linearization—though neither provides a systematic framework for comparative statics.

The Lyapunov method consists of constructing a function that decreases along the trajectory induced by the optimal policy, thereby proving stability of a candidate steady state. This approach applies to a broader class of economies than ours because it does not rely on monotonicity conditions. However, it provides little guidance on how to construct an appropriate Lyapunov function; in this sense, it is essentially an 'example-by-example' method. Moreover, while useful for verifying global stability of a single steady state, it is unclear how to apply it systematically in environments with multiple steady states.

The other general approach relies on linearizing the system around a steady state to assess its local stability. This method has a long tradition in both mathematics (via linearization theorems for dynamical systems) and economics (e.g., Stokey et al. (1989)).

However, it faces several limitations relative to our approach. First, its applicability typically requires additional assumptions, such as strict concavity of the per-period payoff or, at a minimum, that the optimal policy correspondence be a single-valued function. Second, even within this class of problems, a central limitation of linearization is that it guarantees only the existence of a local region of attraction around the steady state, without characterizing the region itself. For applied work, this is a significant drawback: one cannot determine whether a given initial condition lies within that region, and therefore cannot conclude whether convergence to the steady state will actually occur.

Our method is qualitatively different, as illustrated in Figure 1. It establishes a connection between the dynamical properties of steady states—specifically, their stability or instability—and a primitive, simple object: the *locator function*.

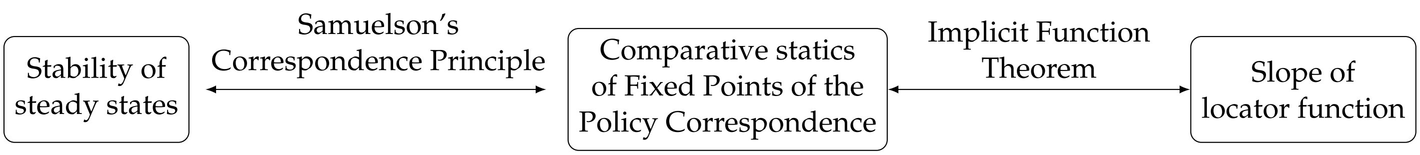


Figure 1: Scheme of our method.

First, Proposition 3.2 below shows that steady states coincide with fixed points of the optimal policy correspondence. We then introduce the *locator function*, a smooth object whose zeros contain these fixed points. We then use this object to study dynamics, relating its slope to the local stability of steady states. The main challenge is that the optimal policy need not be single-valued, let alone differentiable, ruling out standard linearization methods. We address this challenge in two steps.

The first step builds on Samuelson's Correspondence Principle (SCP; Samuelson (1983)), and in particular on its modern formulation due to Echenique (2000, 2002), which extends the SCP to environments where the policy correspondence satisfies suitable monotonicity conditions with respect to a parameter of interest. However, in these formulations the policy correspondence is taken as primitive and the required order properties are imposed directly. In our setting, by contrast, the optimal policy is endogenous, so these properties must be derived rather than assumed. We establish them by applying the Milgrom–Shannon Theorem (Milgrom and Shannon (1994)) and proving that the value function is supermodular in the state–discount-factor space.[1] Combined with Proposition 3.2, these results allow us to connect the stability properties of steady states to the comparative statics of the optimal policy with respect to the discount factor.

The second step connects these comparative statics results to the slope of the locator function. Because the locator function serves as a smooth proxy for the optimal correspondence—more precisely, for the difference between the correspondence and the identity mapping—we can use the Implicit Function Theorem to establish this connection.

At a high level, the present work offers the same type of advantage over standard dynamic programming methods (cf. Stokey et al. (1989)) as Echenique (2000, 2002) and Milgrom and Shannon (1994) offer over the classical Samuelson Correspondence Principle and Topkis' comparative statics, respectively. In both cases, the key step is to dispense with convexity, concavity, and smoothness assumptions and instead base the analysis on monotonicity and order-theoretic conditions. Our contribution follows a similar route in a dynamic optimization setting: rather than relying on the usual regularity conditions underlying recursive methods, we derive results directly from the order structure of the problem.

Roadmap The remainder of the paper proceeds as follows. Section 2 presents the general setup and assumptions. Section 3 introduces the locator function and presents our main result.Section 4 provides refinements of this result under further assumptions. Section 5 presents the learning-by-doing application mentioned above. Section 6 presents some remarks discussing the relationship between our approach and existing methods in the literature as well as how to extend some of our assumptions. Section 7 briefly concludes.

2 Environment

In this section we define the planner's problem. The planner chooses a sequence of the aggregate state, $(s_t)_{t=1}^{\infty}$, given an initial value, s_0, to maximize $\sum_{t=0}^{\infty} \delta^t \pi(s_t, s_{t+1})$ subject to $s_{t+1} \in \Upsilon(s_t)$, where $\delta \in [0,1)$ is the discount factor, $\pi : \mathbb{S}^2 \to \mathbb{R}$ is the per-period payoff,

[1] In fact, we prove a stronger result: the value function is differentiable, and its derivative with respect to the discount factor is non-decreasing in the state.

and $\Upsilon : \mathbb{S} \rightrightarrows \mathbb{R}$ is the constraint correspondence. The state space $\mathbb{S}$ is assumed to be a bounded, convex subset of $\mathbb{R}$.[2] This problem can be cast recursively with the Bellman equation, with $V : \mathbb{S} \times [0, 1) \to \mathbb{R}_+$ as the value function given by

$$V(s, \delta) = \max_{s' \in \Upsilon(s)} \pi(s, s') + \delta V(s', \delta).$$

The optimal policy correspondence is denoted by $\Gamma(\cdot, \delta) : \mathbb{S} \rightrightarrows \mathbb{R}_+$ and is given by

$$s \mapsto \Gamma(s, \delta) := \arg\max_{s' \in \Upsilon(s)} \pi(s, s') + \delta V(s', \delta).$$

We now introduce some technical definitions and list the assumptions used throughout the paper. For any set S, S^o denotes its interior. We say a function $x \mapsto f(x, y)$ is uniformly right differentiable at (x, y) if, for any sequence $(y_n)_n$ converging to y and any positive sequence $(\Delta_n)_n$ converging to zero,

$$\partial_1^+ f(x, y) := \lim_{n\to\infty} \frac{f(x + \Delta_n, y) - f(x, y)}{\Delta_n} \textit{ exists and is continuous at } (x, y) \textit{ and}$$
$$\lim_{n\to\infty} \left| \frac{f(x + \Delta_n, y_n) - f(x, y_n)}{\Delta_n} - \partial_1^+ f(x, y) \right| = 0$$

It is uniformly left differentiable at (x, y) if the same holds but with $(\Delta_n)_n$ being negative and is denoted as $\partial_1^- f(x, y)$. We say $x \mapsto f(x, y)$ is uniformly smooth almost everywhere (a.e.) if it is (uniformly) right and left differentiable at every point and these derivatives coincide *except possibly in a finite set*. We say a function $x \mapsto f(x, y)$ is smooth if it is right and left differentiable at every point and these derivatives coincide *everywhere*.

For the per-period payoff we make the following assumption:

Assumption 1. *π is continuous and (i) $s \mapsto \pi(s, s')$ is increasing, uniformly smooth a.e. with $\partial_1^+ \pi \geq \partial_1^- \pi$; (ii) $s' \mapsto \pi(s, s')$ is smooth with derivative denoted as π_2; (iii) $s \mapsto \pi_2(s, s')$, $s' \mapsto \partial_1^+ \pi(s, s')$, and $s' \mapsto \partial_1^- \pi(s, s')$ are increasing.*

Part (i) requires the per-period payoff to be monotone in the current state, allowing for a finite number of kinks. The restriction on left and right derivatives ensures global

[2] In SM.1 we extend the framework to allow for unbounded state space. Under standard "growth conditions" on the per-period payoff the results for the bounded case continue to hold.

monotonicity as well as monotonicity and differentiability of the value function, even in the absence of concavity. The uniform smoothness is technical and is used to establish differentiability of the value function in non-concave settings.

Crucially, the analysis relies only on monotonicity of π in the current state, not on its direction. While part (i) assumes monotone increase, all results extend, with minor notational changes, to the case in which π is monotone decreasing; see Section 6 for a more detailed discussion.

Part (ii) is standard. Part (iii) imposes complementarity between current and future states: the marginal return to the next-period state is increasing in the current state, which under smoothness reduces to $\pi_{12} > 0$. Together, parts (i) and (iii) can be viewed as replacing the standard strict concavity assumption. While not innocuous, they encompass many economically relevant applications; we refer the reader to Section 6 for further discussion of their role.

We impose the following restrictions on the constraint correspondence:

Assumption 2. *(i) $s \mapsto \Upsilon(s)$ is continuous, convex-/compact-valued, with non-empty interior over $\mathbb{S}^o$; (ii) $s \mapsto \Upsilon(s)$ is non-decreasing in the inclusion sense; (iii) $s \mapsto \Upsilon(s)$ is non-decreasing in the strong set order sense.*[3]

Part (i) is standard. Part (ii) is used solely for establishing monotonicity of the value function. While standard (cf. Stokey et al. (1989) Assumption 4.6) it could be too strong for some applications (e.g., the Neo-Classical Growth (NCG) model below). In view of this, Supplemental Material SM.2.2 provides two alternative approaches for establishing monotonicity of the value function that dispense with this assumption. One uses essentially the same insights as the standard approach but with a weaker assumption. The other approach relies on a completely different argument that hinges on a generalized version of the mean value theorem for a.e. smooth functions — to our knowledge, this approach is novel and might be of independent interest. Part (iii) is used to invoke the celebrated Milgrom-Shannon Theorem (Milgrom and Shannon (1994)), which is in turn used to establish monotonicity of the optimal policy correspondence.

[3]That is, take any $Y \in \Upsilon(s)$ and any $Y' \in \Upsilon(s')$ with $s' > s$, then $\min\{Y, Y'\} \in \Upsilon(s)$ and $\max\{Y, Y'\} \in \Upsilon(s')$.

We now illustrate our assumptions in a generalization of the Neo-Classical Growth (NCG) model.

Example generalized NCG model. Following the seminal paper by Skiba (1978), several papers have extended the classical Ramsey-Cass-Koopmans one-sector growth model to allow for a non-concave production function, f. A typical result in these papers is that there are multiple steady state capital levels.

A strand of this literature considers a case where the production function is smooth but non-concave (e.g., Dechert and Nishimura (2012) and see also Majumdar and Mitra (1982); Kamihigashi and Roy (2007) and references therein). Formally, there exists a level of capital s_I such that $s \mapsto f'(s)$ is increasing at any $s < s_I$ and decreasing at any $s > s_I$. A different strand of the literature considers a more extreme failure of concavity wherein the production function exhibits kinks (see Kamihigashi and Roy (2007)). Formally, there exists a s_I such that f is increasing, and piece-wise concave with s_I being the "kink", i.e., $\partial^- f(s_I) \leq \partial^+ f(s_I)$.

All these cases are encompassed by our framework, with the constraint correspondence given by $s \mapsto \Upsilon(s) := [0, f(s)]$ and the payoff function given by $(s, s') \mapsto \pi(s, s') := u(f(s) - s')$, where u is smooth, increasing and concave. The production function is increasing and smooth almost everywhere. There exists a kink, s_I, such that f'' can either change signs (from positive to negative), or $\partial^- f(s_I) < \partial^+ f(s_I)$ — the standard case where f is everywhere smooth and concave, is obviously also allowed. In principle the state space could be all of $\mathbb{R}_+$, but given the features of the model it is essentially without loss of generality to assume $\mathbb{S} = [0, \bar{s}]$ for some $\bar{s} \geq s_{max}$, where s_{max} is the level of capital for which $f(s_{max}) = s_{max}$, $f'(s_{max}) \leq 1$, and $f(s) < s$ for all $s > s_{max}$.

We now verify our assumptions. Observe that $\partial_1^+ \pi(s, s') = u'(f(s) - s')\partial_1^+ f(s)$ and $\partial_1^- \pi(s, s') = u'(f(s) - s')\partial_1^- f(s)$, which satisfy $\partial_1^- \pi \leq \partial_1^+ \pi$ as $u' > 0$ and $\partial^- f \leq \partial^+ f$. Moreover, it is easy to see that differentiability of π holds in the uniform sense. Finally, $s \mapsto \pi_2(s, s') = -u'(f(s) - s')$ is increasing because f is increasing and $-u'$ is too ($u'' < 0$ by concavity). So, Assumption 1 holds.

We conclude this example with a technical remark about this literature. Dechert and Nishimura (2012) only consider a feasible correspondence of the form $s \mapsto \Upsilon(s) :=$

$[0, f(s)]$, which either implies full depreciation or allows for negative investment.[4] In either case, this assumption rules out cases of interest. Thus, it is fruitful to consider the case $s \mapsto \Upsilon(s) := [(1-d)s, f(s) + (1-d)s]$ which reflects an explicit depreciation rate and imposes that investment be non-negative. The technical issue with this formulation is that Assumption 2(ii) is not met. The only usage of this assumption, however, is to establish monotonicity of the value function, thus it suffices to establish monotonicity by other methods. As it turns out, the condition in Lemma SM.2.3 in the Supplemental Material SM.2.2 is met (see the Remark below the lemma) and the value function can be shown to be increasing. Therefore, our theory could also be applied to the case $s \mapsto \Upsilon(s) := [(1-d)s, f(s) + (1-d)s]$.

3 Analytical Results

This section presents the main results of the paper. Section 3.1 establishes basic properties of the value function and the optimal correspondence, including a differentiability result for the value function that extends existing results in the literature and may be of independent interest. Section 3.2 then presents the main results on the analytical characterization of the steady state and the dynamics.

3.1 Properties of the Value Function and Optimal Correspondence

Our analysis relies on establishing monotonicity and differentiability properties of the value function and the optimal policy correspondence. We establish these results in this subsection. However, due to the potential non-concavity and non-smoothness of π, standard “textbook" arguments do not apply, and alternative technical approaches are required. We refer the reader to Supplemental Material SM.2-SM.3 for details.

Lemma 3.1. *The value function V is continuous, bounded, and increasing as a function of s.*

Proof. See Appendix A. □

[4] Investment can attain negative values if we adjust the production function so that output includes undepreciated capital – i.e., if $\tilde{f}$ is the production function and d is depreciation then we proceed as if the production function were $f(s) = \tilde{f}(s) + (1-d)s$ and there were no depreciation. In this case, if consumption exceeds $\tilde{f}(s)$ then $s' = f(s) - c = \tilde{f}(s) + (1-d)s - c < (1-d)s$. Combined with $s' = i + (1-d)s$, we have $i < 0$.

Lemma 3.2. *The optimal policy correspondence* Γ *is non-empty, compact-valued, UHC, function-like, and non-decreasing in the sense that* $\max \Gamma(s, \delta) \leq \min \Gamma(s', \delta)$ *for any* $s \leq s'$.[5]

Proof. See Appendix A. □

Even though there are no restrictions on concavity and $s \mapsto \pi(s, s')$ is not continuously differentiable, the value function still inherits the smoothness properties of π. As the next proposition shows, the left and right derivatives exist and can be characterized in terms of those of π.

Proposition 3.1. $s \mapsto V(s, \delta)$ *has left and right derivatives, which are*

$$\partial_1^+ V(s, \delta) = \max_{y \in \Gamma(s,\delta)} \partial_1^+ \pi(s, y) \text{ and } \partial_1^- V(s, \delta) = \min_{y \in \Gamma(s,\delta)} \partial_1^- \pi(s, y)$$

for any $s \in \mathbb{S}$ *such that* $\Gamma(s, \delta) \subseteq \Upsilon^o(s)$.[6]

Proof. See Appendix A. □

This result and the fact that $\partial_1^+ \pi \geq \partial_1^- \pi$ imply that V is *everywhere* smooth (i.e., continuously differentiable) over Range Γ.[7]

Corollary 3.1. *For any* $s \in \text{Range}(\Gamma \cap \Upsilon^o)$, $\Gamma(s, \delta)$ *is a singleton and*

$$\partial_1^+ V(s, \delta) = \partial_1^- V(s, \delta) = \partial_1^+ \pi(s, \Gamma(s, \delta)) = \partial_1^- \pi(s, \Gamma(s, \delta)) =: \pi_1(s, s') \; \forall s' \in \Gamma(s, \delta)$$

Proof. See Appendix A. □

This corollary generalizes Theorem 2 by Cotter and Park (2006) (see also Clausen and Strub (2016)) to the case where the per-period payoff is only a.e. differentiable. Indeed, a perhaps surprising feature of this result is that even though π is only *a.e.* smooth, the value function is *everywhere* smooth "along the optimal path" (i.e., for any $s \in$ Range Γ). That is, the value function features stronger smoothness than the primitive payoff. This follows from two key facts: first, the optimality of the value function, and second, the monotonicity assumption 1(i).

[5] A correspondence is function-like if its graph has an empty interior.

[6] For states at the boundary of $\mathbb{S}$ only one of the two derivatives are defined.

[7] For any correspondence, F, Range $F := \{y \colon \exists s \in \mathbb{S},\ s.t.\ F(s) \ni y\}$.

Application to the generalized NCG model. An application of Corollary 3.1 shows that the differences in the optimal paths between the generalized NCG model presented in Section 2 and its "textbook" version are due to non-concavity of the production function, not to the kinks per-se — i.e., kinks do not yield new features above and beyond those offered by the non-concavities. To show this, note that Corollary 3.1 states that optimal paths satisfy:[8]

$$u'(f(s_t) - s_{t+1}) - \delta f'(s_{t+1}) u'(f(s_{t+1}) - s_{t+2}) = 0.$$

This expression is the same regardless of whether π has kinks (as in Kamihigashi and Roy (2007)) or is smooth but non-concave (as in Nishimura et al. (2004)).

3.2 Analytical Results for Optimal Paths

This section introduces our new approach for analyzing the steady states and dynamics of the planner's problem: The locator function. A simple object constructed directly from the model's primitives whose roots identify interior steady states and whose slope determines their local stability. By focusing on this function, we reduce the study of dynamic behavior to the analysis of a tractable, easily computable object.

Optimal paths and their steady states. We first formally define some concepts. An *optimal path with initial condition* $s_0 \in \mathbb{S}$ is a mapping $\phi(., s_0) : \mathbb{N}_0 \to \mathbb{S}$ such that

$$\phi(t, s_0) \in \Gamma(\phi(t-1, s_0), \delta),\ \forall t \geq 1,$$

and $\phi(0, s_0) = s_0$. Let $\Phi(s_0)$ be the class of all optimal paths with initial condition s_0.

A first step in our analysis is to characterize the limit points of optimal paths. An obvious candidate for characterizing such behavior is the set of fixed points of $\Gamma(\cdot, \delta)$,

$$\mathcal{R}[\bar{\Gamma}] := \{s \in \mathbb{S} \colon \bar{\Gamma}(s, \delta) = 0\},\ \text{where}\ s \mapsto \bar{\Gamma}(s, \delta) := \Gamma(s, \delta) - s,$$

but in principle there could be other, more complex, candidates such as cycles. The next

[8]Optimal paths are formally defined below, but essentially are $(s_t)_t$ such that $s_{t+1} \in \Gamma(s_t, \delta)$.

proposition confirms this is not the case and that fixed points fully describe the asymptotic behavior of optimal paths.

Proposition 3.2. *For any $s_0 \in \mathbb{S}$ and any $\phi(., s_0) \in \Phi(s_0)$, $\lim_{t\to\infty} \phi(t, s_0) \in \mathcal{R}[\bar{\Gamma}] \cup \partial\mathbb{S}$.*

Proof. See Appendix B. □

This result shows that for any initial condition the limit of the path is well-defined and is a fixed point. However, this limit may not be unique. That is, for a given initial condition, there might be more than one optimal path, and thus more than one limit. This occurs if (and only if) the initial condition, s_0, is a so-called Skiba point, wherein $\Gamma(s_0, \delta) = \{\gamma_l(s_0), \gamma_h(s_0)\}$ such that $\gamma_l(s_0) < s_0 < \gamma_h(s_0)$; e.g., see Figure 4(c) below.

Genericity of Steady States. In light of Proposition 3.2 we henceforth refer to the fixed points of $\Gamma(\cdot, \delta)$ as *steady states*. Throughout the analysis we focus on steady states that are *generic*. Intuitively, non-generic steady states are those that would vanish under small perturbations of Γ — in particular of the discount factor. Geometrically, this genericity restriction rules out steady states for which Γ is tangent to but does not cross the 45° line, or for which Γ, when perturbed, develops a discontinuity.[9]

While every steady state is a fixed point of $\Gamma(\cdot, \delta)$, this set may be too broad to characterize asymptotic behavior: not all fixed points can be reachable, in the sense that they arise as limits of optimal paths that do not start at the fixed point itself. To sharpen our analysis of steady states, we therefore introduce notions of stability and instability.

Stability of steady states. We say that a steady state $e \in \mathcal{R}[\bar{\Gamma}]$ is *stable* if there exists an open interval containing it such that, for any s_0 in the interval and any optimal path $\phi(\cdot, s_0) \in \Phi(s_0)$, $\lim_{t\to\infty} \phi(t, s_0) = e$. The largest such interval is denoted as $\mathcal{B}(e, \delta)$ and will be referred to as the *basin of attraction* of e.[10] We say e is *unstable* if no such open

[9]Formally, a steady state s is generic at δ if there exists an open neighborhood around δ and a unique function q over it such that $q(\delta) = s$ and for any δ' in the neighborhood, $q(\delta')$ is a steady state at δ'. In other words, genericity implies the existence of a local branch through s at δ without bifurcations. This assumption is tightly related to the standard assumption of regularity in general equilibrium theory and game theory Debreu (1970); Dechert and Nishimura (2012) Ch. 17, and hyperbolicity in dynamical systems. To see where this genericity restriction is used, we refer the reader to the proof of Lemma 3.5 below.

[10]In principle, focusing on intervals might restrict the notion of basin of attraction since the latter could be, say, a union of intervals. In our framework, thanks to the monotonicity properties of Γ, this turns out not to be the case: the basin of attraction will indeed be an interval.

interval exists — or, with a slight abuse of notation, $\mathcal{B}(e,\delta)=\{e\}$.

A direct approach to study stability of steady states would be to study the mapping $s \mapsto \bar{\Gamma}(s,\delta) := \Gamma(s,\delta) - s$ and analyze its zeros: where it crosses zero from above, the steady state is stable; where it crosses from below, it is unstable.[11] At this level of generality, however, this result by itself is of limited utility as Γ is not a primitive but rather an outcome of a dynamic programming problem.

The locator function. We propose an alternative approach that relies on a simple function to locate the steady states and provide information about their stability or instability. The proposed function, $\mathcal{L}(\cdot,\delta) : \mathbb{S}^o \to \mathbb{R}$ for any $\delta \in [0,1)$, is dubbed the *locator function* and is given by [12]

$$(s,\delta) \mapsto \mathcal{L}(s,\delta) := \pi_2(s,s) + \delta\pi_1(s,s), \; a.e. \tag{1}$$

The usefulness of the locator function lies on locating steady states and classifying whether they are stable or unstable. To understand why this is the case, note that any interior optimal path, $\phi(\cdot,s_0) \in \Phi(s_0)$, must satisfy the first-order condition (i.e., Euler equation),

$$\pi_2(\phi(t-1,s_0),\phi(t,s_0)) + \delta\pi_1(\phi(t,s_0),\phi(t+1,s_0)) = 0 \quad \forall t \geq 1. \tag{2}$$

Taking limits, it is then clear that interior steady states must be roots of the locator function.[13] The perhaps surprising result is that the locator function also contains information about the stability properties of the steady state, as we show below.

To proceed, we impose the following technical condition:

Assumption 3. *All roots of $s \mapsto \mathcal{L}(s,\delta)$ are well-separated and regular.*[14]

Regularity is a necessary condition for applying the Implicit Function Theorem below

[11] This claim is formally proven in Proposition SM.3.1 in the Supplemental Material SM.3.3.

[12] The almost everywhere is needed because π_1 may only be defined a.e.. However, $\mathcal{L}$ is only used in neighborhoods of fixed points, for which, by Corollary 3.1, π_1 is well-defined.

[13] The validity of expression 2 follows because, by definition, $\phi_\delta(t,s_0) \in \Gamma(\phi_\delta(t-1,s_0),\delta)$ and is interior. So, all the assumptions of Proposition 3.1 and Corollary 3.1 are met.

[14] A root, r, is said to be regular if $\mathcal{L}_1(r,\delta) \neq 0$; it is well-separated if there exists a $c > 0$ such that $|r-r'| \geq c$ for any distinct roots $r \neq r'$.

which is key for the proof of the theorem. It is a generic property in the sense that if a root fails to be regular for some value of δ, then small perturbations of δ will eliminate that root. The assumption of well-separated roots serves mainly to rule out pathological cases in which the locator function exhibits infinitely frequent oscillations. Lemma SM.4.1 in the Supplemental Material SM.4 provides sufficient conditions, based on the derivatives of the locator function, for this to hold.

Main result. The following theorem shows that the locator function can be used not only to identify interior steady states, but also to assess their local stability by examining the sign of its derivative at interior roots: a negative derivative implies stability, while a positive one implies instability. Henceforth, a steady state is *interior* if it belongs to $\mathcal{R}^o[\bar{\Gamma}] := \{s \in \mathcal{R}[\bar{\Gamma}] \colon s \in \Upsilon^o(s)\}$.[15]

Theorem 1. *For any interior steady state s, the following are true:*

1. $s \in \mathcal{R}^o[\mathcal{L}]$.
2. *If* $\mathcal{L}_1(s, \delta) < 0$, *then s is stable.*
3. *If* $\mathcal{L}_1(s, \delta) > 0$, *then s is unstable.*

By providing a link between the (local) monotonicity of the locator function and the stability or instability of steady states, Theorem 1 offers a simple way of identifying the limit points of the planner's dynamics: for stable fixed points the locator function cuts zero "from above." Importantly, unlike Γ, which is the solution of a dynamic programming problem, the locator function depends on primitives and is relatively easy to compute and study.

The theorem does not establish an equivalence between steady states and roots of the locator function: while every interior steady state is a root of the locator function, the converse does not hold. This is not surprising as it reflects the fact that first-order conditions are not sufficient in the absence of concavity. This raises a natural question: under what conditions on π does the locator function avoid generating "false positives"? We explore this question further in Section 4.

[15] $\mathcal{R}^o[\mathcal{L}]$ is defined analogously.

What might be surprising, however, is that the locator function does more than characterize interior steady states as roots. Theorem 1 shows that the sign of its derivative locally determines whether a steady state is stable or unstable; the heuristics behind this result are presented below. Moreover, this classification holds even when the mapping $s \mapsto \pi(s, s')$ is not differentiable everywhere. This last result follows because the first-order condition (2) is identical to the one that would arise under smoothness. Thus, under our assumptions — particularly the monotonicity conditions in Assumption 1 — potential "kinks" in π do not affect the qualitative properties of optimal paths.

Application to the generalized NCG model. We now return to the generalized NCG model to illustrate the usefulness of Theorem 1 and the locator function. The dynamics of this economy were previously analyzed in Majumdar and Mitra (1982) and Dechert and Nishimura (2012). Our analysis offers an alternative and complementary perspective, showing in passing how the locator function helps explain the different cases presented in Dechert and Nishimura (2012). For expositional purposes, we assume that f is smooth and that f'' changes signs at only one point, s_I; if more points like s_I exist we simply repeat the analysis in each interval.

The locator function is given by

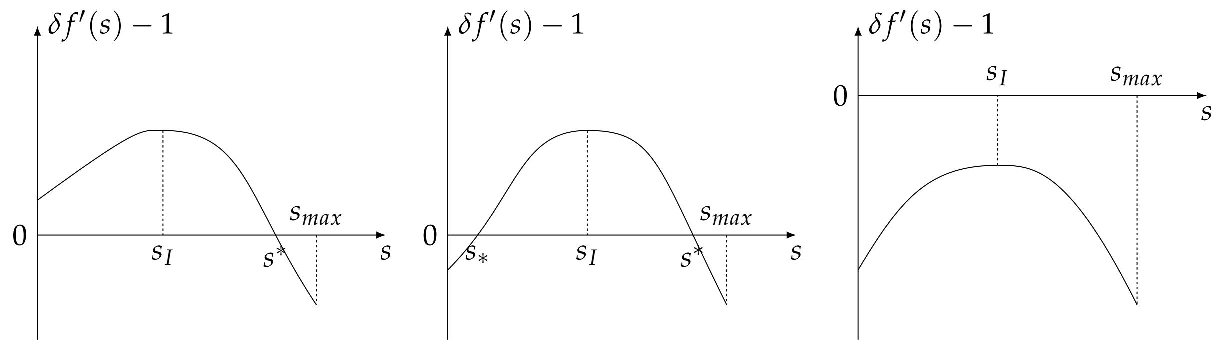
$$\mathcal{L}(s, \delta) = u'(f(s) - s)(\delta f'(s) - 1),$$

and $\mathbb{S} := [0, s_{max}]$, where s_{max} is such that $\mathcal{L}(s_{max}, \delta) < 0$. Since $u' > 0$, the only interior roots are those of function $s \mapsto \delta f'(s) - 1$. Since $s \mapsto f'(s)$ is increasing in $[0, s_I)$ and decreasing in $(s_I, s_{max}]$, there are at most two interior roots of $s \mapsto \mathcal{L}(s, \delta)$, which we denote as $s_* < s^*$.

There are three cases: (1) $\delta f'(0) - 1 \geq 0$, which is dubbed as mild discounting by Dechert and Nishimura (2012); (2) $\delta f'(0) - 1 < 0 < \delta f'(s_I) - 1$; and (3) $\delta f'(s_I) < 1$.

It is easy to see by Figure 2(3) that in the last case there are no roots of the locator function. Consequently, by Theorem 1, there are no interior steady states and the only steady state is $s = 0$, which is globally stable.[16] In the other two cases, the locator function

[16]Since $\delta f'(s_{max}) < 1$, this implies that $\Gamma(s_{max}, \delta) < s_{max}$. Since there are no roots, then the only steady state is $s = 0$ and is globally stable.



(1) One (stable) root. (2) Two (unstable and stable) roots. (3) No roots.

Figure 2: $\frac{\mathcal{L}(s,\delta)}{u'(f(s)-s)}$ in generalized NCG model

has either a single interior root, s^*, as in Figure 2(1), or two interior roots, s_* and s^*, as in Figure 2(2). By Theorem 1, s^* is the only candidate for a locally stable interior steady state, but in case (2) we could also have a locally unstable steady state at s_*.

Below we provide a more refined analysis of these cases, but we hope this example illustrates a key advantage of our approach: examining the shape of the locator function involves significantly simpler calculations while delivering results comparable to those obtained through standard methods. In addition, we can extend the non-concave Neoclassical Growth Model to accommodate any production function, not just those with kinks or s-shaped profiles. Indeed, Theorem 1 implies that for any function f, the stable steady states must satisfy two conditions: (a) f' equals $1/\delta$, and (b) f is concave at that point. This generalizes the classical results for the Neoclassical Growth Model to a much broader class of settings.

Proof of Theorem 1. Throughout this section it is useful to make explicit the dependence of roots and fixed points on the discount factor. Hence, we use $\mathcal{R}^o[\bar{\Gamma}(\cdot,\delta)]$ and $\mathcal{R}^o[\mathcal{L}(\cdot,\delta)]$. Also, we define $\mathcal{E} = \{(s,\delta) \in \mathbb{S} \times [0,1) : \bar{\Gamma}(s,\delta) = 0\}$.

The first part of Theorem 1 follows from the following lemma.

Lemma 3.3. *For any $\delta \in [0,1)$, $\mathcal{R}^o[\bar{\Gamma}(\cdot,\delta)] \subseteq \mathcal{R}^o[\mathcal{L}(\cdot,\delta)]$.*

Proof. See Appendix D. □

The proof of the second part of the theorem rests on a two-step argument. The first step links the stability of a steady state to the comparative statics of fixed points of the

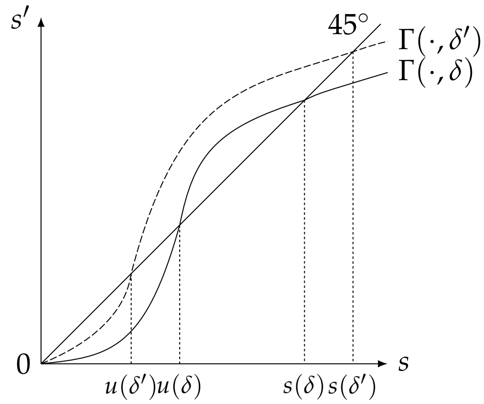


Figure 3: Comparative statics of stable and unstable fixed points

optimal correspondence with respect to the discount factor. The second step relates these comparative statics to the slope of the locator function evaluated at the root corresponding to the steady state.

In the first step, stability is inferred from the response of steady states to small changes in the discount factor. The first building block for this is a comparative static result showing that the mapping $\delta \mapsto \Gamma(s,\delta)$ is non-decreasing.

Lemma 3.4. *For any $\delta \leq \delta'$, $\max \Gamma(s,\delta) \leq \min \Gamma(s,\delta')$.*

Proof. See Appendix D. □

Intuitively, as the discount factor rises the planner attaches greater weight to future payoffs, and since $s \mapsto V(s,\delta)$ is increasing, the planner is drawn toward higher values of the next-period state. As illustrated in Figure 3, when the policy correspondence weakly shifts up with δ, stable fixed points cannot shift left as δ increases, while unstable ones cannot shift right. The following lemma formalizes this observation.

Lemma 3.5. *For any $(e,\delta) \in Graph\mathcal{E}^o$, consider an open neighborhood around δ, $U_q(\delta)$, and a continuous mapping q from $U_q(\delta)$ to $\mathbb{R}$ such that $q(\delta) = e$ and $q(\delta') \in \mathcal{R}^o[\bar{\Gamma}(\cdot,\delta')]$ for any $\delta' \in U_q(\delta)$. Then q is non-decreasing at δ if e is stable and non-increasing if e is unstable.*

Proof. See Appendix D. □

In the second step, we connect comparative statics with respect to the discount factor to the slope of the locator function at the corresponding root. By Lemma 3.3, any mapping

q in the previous lemma satisfies $\mathcal{L}(q(\delta'), \delta') = 0$ for any $\delta' \in U_q(\delta)$.[17] By Corollary 3.1, $\mathcal{L}$ is continuously differentiable around each $s \in \mathcal{R}^o[\bar{\Gamma}(\cdot, \delta)]$, and $\mathcal{L}_1(s, \delta) \neq 0$ by Assumption 3. Hence, by the Implicit Function Theorem, there exists a neighborhood of δ contained in $U_q(\delta)$ such that

$$\frac{dq(\delta)}{d\delta} = -\frac{\pi_1(q(\delta), q(\delta))}{\mathcal{L}_1(q(\delta), \delta)}. \tag{3}$$

Since $\pi_1 > 0$ by Assumption 1, q is non-decreasing iff $\mathcal{L}_1(q(\delta), \delta) < 0$ and non-increasing iff $\mathcal{L}_1(q(\delta), \delta) > 0$. Thus, if $e = q(\delta)$ is stable, then, by Lemma 3.5, q must be non-decreasing at δ, so $\mathcal{L}_1(e, \delta) < 0$; and if $e = q(\delta)$ is unstable, then, by Lemma 3.5, q must be non-increasing, so $\mathcal{L}_1(e, \delta) > 0$. Since stability and instability are mutually exclusive, we conclude that $\mathcal{L}_1(e, \delta) < 0$ implies stability of e and $\mathcal{L}_1(e, \delta) > 0$ implies instability of e, as claimed in the second part of Theorem 1.

We conclude this section with a remark. The first step of the proof — the link between stability and comparative statics — is related to Samuelson's Correspondence Principle (Samuelson (1983); see also Echenique (2002)). Samuelson's principle uses local dynamic stability (e.g., under tâtonnement dynamics) to sign comparative statics, such as concluding that equilibrium price rises with an outward demand shift because the demand curve must be flatter than the supply curve at a locally stable equilibrium. Our approach runs in the opposite direction: we use comparative statics with respect to the discount factor to infer local stability of the model's underlying dynamics.

4 Refinements and Implications of Theorem 1

While Theorem 1 provides a general method for identifying and classifying the stability of interior steady states using the locator function, this section highlights important cases in which additional analysis yields sharper insights. We begin by showing that when the locator function crosses zero only once and from above, the unique interior root corresponds to a globally stable steady state. We then examine the case in which the locator function is inverted-U shaped with two interior roots, demonstrating that meaningful

[17] Existence of the neighborhood $U_q(\delta)$ and the mapping q is ensured by our assumption of restricting the analysis to generic points.

conclusions about stability can still be drawn. Next, we consider the special case of a strictly concave per-period payoff function and show that, under an additional condition, the locator function fully characterizes all interior steady states *and their basins of attraction*, thereby providing a complete characterization of the dynamics of optimal paths. Finally, we show how the locator function can be used to do comparative statics for stable steady states. Together, these results clarify the scope and limitations of the main theorem and offer guidance for its application across a range of economic environments.

Throughout this section, to facilitate the exposition of the results, we maintain the assumption of smoothness of the locator function and that $s \in \Upsilon^o(s)$ for all $s \in \mathbb{S}^o$.

4.1 Locator function is single-crossing from above

In some applications, it is useful to impose conditions that ensure the existence of a unique globally stable steady state. This subsection provides a sufficient condition on the locator function under which it identifies such a steady state. Specifically, we require that the locator function satisfy a single-crossing property from above — i.e., there exists a point $c \in \mathbb{S}^o$ such that $\mathcal{L}(s,\delta) > 0$ for all $s < c$, $\mathcal{L}(s,\delta) < 0$ for all $s > c$, and $\mathcal{L}(c,\delta) = 0$.

Proposition 4.1. *Suppose the locator function satisfies single-crossing from above. Then there exist a unique interior steady state, given by the root of the locator function, and it is globally stable over $\mathbb{S}^o$.*[18]

Proof. See Appendix C.1. □

Application to the generalized NCG model. Consider case (1) above, in which $\delta f'(0) > 1$. This inequality implies that $\mathcal{L}(0,\delta) > 0$, and since $s \mapsto f'(s)$ is increasing on $[0, s_I)$ and decreasing on $(s_I, s_{\max}]$, it follows that $s \mapsto \mathcal{L}(s,\delta)$ is single-crossing decreasing from above. Thus, by Proposition 4.1, the root s^* is the unique interior steady state and is globally stable over $\mathbb{S}^o$.

[18] The qualifier "over $\mathbb{S}^o$" means that its basin of attraction is the whole of $\mathbb{S}^o$. That is, the result does not rule out *unstable* steady states at the boundary of $\mathbb{S}$.

4.2 Locator function has two interior roots

In some situations the locator function will have multiple roots and thus Proposition 4.1 cannot be used. The next proposition shows that the locator function can still identify stable interior steady states in this case.

Proposition 4.2. *Suppose the locator function has two roots, $s_* < s^*$ and $\mathcal{L}(\underline{s}, \delta), \mathcal{L}(\bar{s}, \delta) < 0$.*[19] *Then either*

1. *$\underline{s}$ is the unique globally stable steady state, or*

2. *s^* is the only locally stable interior steady state.*

Moreover, if $s' \mapsto \pi(s, s')$ is strictly concave and there exists $s \in \mathbb{S}^o$ such that $\pi_2(s, s) = 0$, then case 2 is the only possibility.

Proof. See Appendix C.2. □

Given that the locator function has exactly two roots, $s_* < s^*$, and is negative at both boundaries, Theorem 1 implies that either $\Gamma(s, \delta) < s$ for all $s \in \mathbb{S}^o$, or Γ eventually crosses the 45° line. In the first case, all trajectories converge to the lower boundary $\underline{s}$, which is then the unique globally stable steady state. In the second case, moving from right to left, the first crossing from above must occur at s^* , which must then correspond to a locally stable steady state. There are two possible sub-cases for the behavior of $\Gamma(\cdot, \delta)$ on $(\underline{s}, s^*)$: (i) it remains above the 45° line; (ii) it crosses the 45° line, in which case Theorem 1 implies the crossing occurs at s_*.[20] In all cases, s^* remains the only locally stable interior steady state.

Finally, the added condition that $s \mapsto \pi(s, s')$ is strictly concave and satisfies $\pi_2(s, s) = 0$ at some interior point ensures that the myopic planner ($\delta = 0$) has an interior steady state. But since $\delta \mapsto \Gamma(s, \delta)$ is non-decreasing (see Lemma 3.2), this implies that $\Gamma(s, \delta) \geq s$ somewhere, contradicting the possibility of Γ being strictly below the 45° line throughout and ruling out $\underline{s}$ as a globally stable steady state.

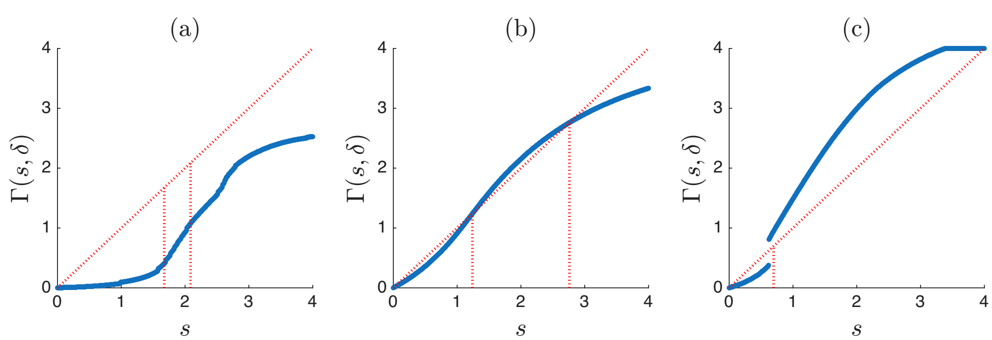
[19] Here and throughout, $\underline{s}$ and $\bar{s}$ are the lower and upper bounds of the state space. The case where $\mathcal{L}(\underline{s}, \delta), \mathcal{L}(\bar{s}, \delta) > 0$ is completely analogous and thus omitted.

[20] Observe that the optimal correspondence cannot "jump down" because it is non-decreasing (Lemma 3.2) and cannot cross the 45° line more than twice as the locator function has only two roots.

Application to the generalized NCG model. Proposition 4.2 can be applied to study case (2) in the generalized NCG model, in which $\delta f'(0) - 1 < 0 < \delta f'(s_I) - 1$. Since $s \mapsto f'(s)$ is increasing in $[0, s_I)$ and decreasing in $(s_I, s_{max}]$, the locator function is indeed inverted-U shaped, and since $\delta f'(0) < 1$ and $\delta f'(s_{max}) < \delta f'(s^*) = 1$, then $\mathcal{L}(0,\delta) < 0$ and $\mathcal{L}(s_{max}, \delta) < 0$. Thus, by Proposition 4.2 either 0 is the unique globally stable steady state or s^* is the only locally stable interior steady state. As it turns out, both cases are possible, and depend on the parametrization. To see this, assume CRRA preferences, $c \mapsto u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ with $\gamma \neq 1$, and a convex-concave production function given by $s \mapsto f(s) = -\frac{a}{3}s^3 + \frac{b}{2}s^2 + cs$. Varying the values of the different parameters, Figure 4(a) shows a case in which $\underline{s}$ is the unique globally stable steady state, whereas Figures 4(b) and (c) show that s^* is the only locally stable interior steady state.

Interestingly, Figure 4(c) illustrates a setting in which the locator function has two roots, but only one corresponds to a steady state. The other root is not a steady state and instead arises from the presence of the Skiba point. This example underscores that Theorem 1 is, in a sense, sharp: the inclusion of steady states among the roots of $\mathcal{L}$ cannot generally be strengthened — except in special cases, such as those covered in Propositions 4.1 and 4.3 below.

Figure 4: Neoclassical growth model with inverted-U shaped locator function



Note: (a) with $a = 0.266, b = 1, c = 0.5, \delta = 0.7$, (b) with $a = 0.25, b = 1, c = 4.7, \delta = 0.18$, (c) with $a = 0.2, b = 1, c = 1.4, \delta = 0.5$. Relative risk aversion coefficient: $\gamma = 1.7$. Vertical dashed lines show the roots of their corresponding locator functions.

4.3 Strictly concave per-period payoff function

In this subsection, we focus on the case in which π is strictly concave, which is standard in the literature (see, e.g., Stokey et al. (1989)) and requires $\pi_{11}\pi_{22} > (\pi_{12})^2$. This condition ensures that cross effects are small relative to the curvature of π in s and s', so that the overall payoff remains strictly concave.[21]

The next result shows that in this setting, under mild additional conditions, the locator function not only characterizes stable fixed points, but also their basin of attraction. To state the result we need the following definition of a basin of attraction of a point $e \in \mathbb{S}$ under a function $F : \mathbb{S} \to \mathbb{R}$:

$$\begin{aligned} \mathcal{B}[F](e) &:= \mathcal{B}^{-}[F](e) \cup \mathcal{B}^{+}[F](e), \\ \text{with } \mathcal{B}^{+}[F](e) &:= \{s \in \mathbb{S} \colon s < e \text{ and } \forall s' \in (s,e)\ F(s') > 0\} \cup \{e\} \\ \text{and } \mathcal{B}^{-}[F](e) &:= \{s \in \mathbb{S} \colon s > e \text{ and } \forall s' \in (e,s)\ F(s') < 0\} \cup \{e\}. \end{aligned}$$

That is, $\mathcal{B}^{+}[F](e)$ is the set of all points $s \in \mathbb{S}$ such that for any point between s and e, the function $s \mapsto F(s)$ is positive — this set includes e as a convention that simplifies the exposition. We refer to $\mathcal{B}[F](e) = \mathcal{B}^{-}[F](e) \cup \mathcal{B}^{+}[F](e)$ as the *Basin of attraction (BoA) for F* — this terminology is justified by the following result.

Proposition 4.3. *Suppose π is strictly concave and satisfies*

$$\max_{a \in \Upsilon(s)\colon a > s} \{\pi_2(s,a) + \delta \pi_1(s,a)\} \leq 0 \leq \min_{a \in \Upsilon(s)\colon a < s} \{\pi_2(s,a) + \delta \pi_1(s,a)\}, \ \forall s \in \mathcal{R}^o[\mathcal{L}]. \quad (4)$$

Then $\mathcal{R}^o[\bar{\Gamma}] = \mathcal{R}^o[\mathcal{L}]$ and $\mathcal{B}(e,\delta) = \mathcal{B}[\bar{\Gamma}](e) = \mathcal{B}[\mathcal{L}](e)$, $\forall e \in \mathcal{R}^o[\bar{\Gamma}]$.

Proof. See Appendix C.3. □

This result implies that when π is strictly concave (and satisfies condition 4) the locator function identifies not only the steady states but also their basins of attraction thereby giving an easy-to-use tool for understanding the dynamics of optimal paths. This is summarized in Table 1

Concepts	Relationship	Γ	Relationship	$\mathcal{L}$
Steady State	= (Proposition 3.2)	$\mathcal{R}[\bar{\Gamma}]$	= (Proposition 4.3)	$\mathcal{R}[\mathcal{L}]$
Basin of Attraction ($\mathcal{B}(\cdot,\delta)$)	= (Proposition SM.3.1)	$\mathcal{B}[\bar{\Gamma}](\cdot)$	= (Proposition 4.3)	$\mathcal{B}[\mathcal{L}](\cdot)$

Table 1: Equivalences under strict concavity of π

[21] The condition $\pi_{11}\pi_{22} \geq (\pi_{12})^2$ follows from the fact that the Hessian has to have non-negative determinant.

Condition 4 ensures that the locator function will never deliver "false fixed points". To illustrate this, suppose there exists an $s \in \mathcal{R}^o[\mathcal{L}]$ but $\Gamma(s,\delta) > s$. Since $s' \mapsto V_1(s',\delta)$ is decreasing (by strict concavity of the value function), the envelope condition implies $\pi_1(s,\Gamma(s,\delta)) > \pi_1(\Gamma(s,\delta),\Gamma^2(s,\delta))$, and hence

$$\pi_2(s,\Gamma(s,\delta)) + \delta\pi_1(s,\Gamma(s,\delta)) > \pi_2(s,\Gamma(s,\delta)) + \delta\pi_1(\Gamma(s,\delta),\Gamma^2(s,\delta)) = 0.$$

But from the first inequality in condition 4 we have $\pi_2(s,\Gamma(s,\delta)) + \delta\pi_1(s,\Gamma(s,\delta)) \leq 0$, yielding a contradiction. An analogous result holds for the case $\Gamma(s,\delta) < s$, thereby showing that if s is such that $\Gamma(s,\delta) \neq s$, then $\mathcal{L}(s,\delta) \neq 0$.

Condition 4 appears somewhat cumbersome, but is easy to verify. For instance, it holds if $\text{sign}\{\mathcal{L}(s,\delta)\} = \text{sign}\{\pi_2(s,a) + \delta\pi_1(s,a)\}$ for any $a \in \Upsilon(s)$. In the generalized NCG model, $\pi_2(s,a) + \delta\pi_1(s,a) = u'(f(s)-a)(\delta f'(s) - 1)$ and since $u' > 0$, the sign of this function is determined by the sign of $\delta f'(s) - 1$, which precisely determines the sign of the locator function. Thus Condition 4 holds. Additional sufficient conditions are provided by Lemma SM.6.1 in the Supplemental Material SM.6 — for instance it shows that Condition 4 is implied by $\pi_{22} + \delta\pi_{12} \leq 0$.

4.4 Using the Locator Function for Comparative Statics

Suppose the per-period payoff is parameterized by an index $\xi \in \mathbb{R}$, against which we would like to perform comparative statics of steady states.[22] We focus on *stable* steady states, as unstable ones will never be attained (unless in trivial cases where the initial state is the steady state itself). The next result shows that the locator function can be used to perform comparative statics of steady states against the parameter ξ.

To show this, we slightly change notation and allow the locator function to depend explicit on ξ by using $\mathcal{L}(\cdot,\delta,\xi)$. Suppose $(s,\xi) \mapsto \mathcal{L}(s,\delta,\xi)$ is continuously differentiable. By Theorem 1, for any ξ and any stable interior steady state, $s(\xi)$, $\mathcal{L}(s(\xi),\delta,\xi) = 0$ and $\mathcal{L}_1(s(\xi),\delta,\xi) < 0$. Thus, by the IFT, the mapping $\xi \mapsto s(\xi)$ is continuously differentiable with derivative given by $s'(\xi) = -\frac{\mathcal{L}_3(s(\xi),\delta,\xi)}{\mathcal{L}_1(s(\xi),\delta,\xi)}$, with the sign of $s'(\xi)$ equal to the sign of $\mathcal{L}_3(s(\xi),\delta,\xi)$. This logic implies the following result

[22]The results here easily extend to $\xi \in \mathbb{R}^q$ for $q > 1$.

Proposition 4.4. *Suppose $(s,\xi) \mapsto \mathcal{L}(s,\delta,\xi)$ is continuously differentiable. Then for any $\xi \in \mathbb{R}$, any stable interior steady state, $s(\xi)$, is increasing (decreasing) in ξ iff $\mathcal{L}_3(s(\xi),\delta,\xi)$ is positive (negative).*

5 Application: Intertemporal Economies of Scale

This application considers a planner's problem in a two-sector economy where one sector exhibits learning-by-doing, generating a trade-off between present consumption and future productivity gains. Depending on the strength of learning by doing, the planner's optimal path may have a globally stable interior steady state, or instead exhibit path dependence with multiple steady states.

This application is motivated by a large literature, including Krugman (1987), Lucas Jr (1988), Young (1991), and Redding (1999), which study how learning-by-doing externalities can give rise to multiple steady states and inefficient specialization in market economies absent industrial policies. Closer to our analysis, Bardhan (1971) and Melitz (2005) derive necessary conditions for optimality but, through strict concavity (Bardhan) or bounded learning that collapses the problem to a binary protect-or-not decision (Melitz), stop short of the multiple interior steady states our approach characterizes.

We study an open economy, Home, with two goods, and a unit of labor supplied inelastically. We use s and s' to denote employment in production of good 1 in the previous and current periods. There is learning by doing in the production of good 1, with this period's productivity shifter a function of last period's employment. Specifically, production of good 1 in the current period is given by $H(s)F(s')$, where $H(s) := s^\theta$ and $F(s') := (s')^\alpha$ with $\theta, \alpha \in (0,1)$. Production of good 0 in the current period is given by $G(1-s')$, with $G : \mathbb{R}_+ \to \mathbb{R}$ smooth, increasing and concave.

To simplify the analysis, we further assume that Home's per-period utility function $U : \mathbb{R}^2_+ \to \mathbb{R}$ is quasilinear, $U(c_0, c_1) = c_0 + u(c_1)$, with $u : \mathbb{R}_+ \to \mathbb{R}$ smooth, increasing and strictly concave. Home is the only producer of good 1, with the rest of the world's inverse demand curve for Home's exports of good 1 given by $e \mapsto p(e) = be^{-1/\varepsilon}$. Using good 0 as numeraire, and suppressing subindex 1 for good 1 consumption, the per-period payoff function of Home's planner is then

$$(s,s') \mapsto \pi(s,s') := \max_c u(c) + G(1-s') + p\left(H(s)F(s') - c\right) \cdot \left(H(s)F(s') - c\right).$$

The only restriction on the current period's employment in production of good 1 is that it respect the resource constraint on labor, hence $s' \in \Upsilon(s)$ with $s \mapsto \Upsilon(s) := [0,1]$. Home's production of good 1 is $H(s)F(s')$, of which $e(s,s') := H(s)F(s') - c(s,s')$ units are exported at price $p\left(e(s,s')\right)$, with $c(s,s')$ denoting the solution of the above optimization problem for a given (s,s') pair. Export revenues fund imports of good 0, complementing Home's own production of that good for consumption.[23]

Verification of Assumptions 1-2. Since the planner behaves as a monopolist for good 1 in the foreign market, we need to assume the elasticity of foreign demand for Home's exports of good 1, $\varepsilon = -\left(\frac{d \ln p(e)}{d \ln e}\right)^{-1}$, is larger than 1 for there to be an interior solution. In the following lemma, we show that as long as the elasticity of marginal utility of good 1, $c \mapsto \psi(c) := -\left(\frac{d \ln u'(c)}{d \ln c}\right)^{-1}$, is always larger than 1, Assumption 1 holds.

Lemma 5.1. *If $\varepsilon > 1$ and $\psi(\cdot) > 1$, then Assumption 1 is satisfied.*

Proof. See Appendix D.1. □

To gain some intuition for this condition, note that by the Envelope Theorem we have $\pi_1(s,s') = u'(c(s,s'))H'(s)F(s')$. An increase in s' implies a higher $F(s')$ and hence a higher $\pi_1(s,s')$. This positive direct effect is counteracted by a negative indirect effect arising from the fact that a higher $F(s')$ implies more consumption of good 1 and hence a lower marginal utility $u'(c)$. In a closed economy $\psi > 1$ is necessary and sufficient to

[23] We can generalize several of these assumptions. First, we can allow for there to be a foreign variety of good 1 (as in the standard Armington trade model) and have Home's utility given by $U(c_0, c_1, c_1^*) = c_0 + u\left(c_1\right) + u^*\left(c_1^*\right)$, where c_1^* is Home's consumption of the foreign variety of good 1 and $u^* : \mathbb{R}_+ \to \mathbb{R}$ is smooth, increasing and strictly concave. Assuming that the relative price between the foreign variety of good 0 and good 1 is fixed, the results below carry through without modification. Alternatively, we can remove the separability assumption between goods 0 and 1 and assume that Home is a small open economy (i.e., the relative price at which it can trade these two goods is fixed), or that it is a closed economy with $U(c_0, c_1) = \frac{\psi}{\psi-1}\left(c_0^{\frac{\sigma-1}{\sigma}} + c_1^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\psi-1}{\psi}\frac{\sigma}{\sigma-1}}$, $1 < \psi < \sigma$. These results are discussed in the Supplemental Material SM.7. What is difficult is to simultaneously allow for non-separability in preferences with Home open to trade with endogenous foreign prices. This is because then $\pi_{12}(s,s') > 0$ is hard to verify for the whole state space.

ensure that the positive direct effect dominates the indirect negative effect. In an open economy this is relaxed because part of the extra output is exported, hence marginal utility declines less. The proof of the lemma shows

$$\pi_{12}(s,s') = u'(c(s,s'))H'(s)F'(s')\frac{\psi(c(s,s')) - 1 + (\varepsilon - 1)\, e(s,s')/c(s,s')}{1 + (\varepsilon/\psi(c(s,s')))\, e(s,s')/c(s,s')},$$

hence $\pi_{12}(s,s') > 0$ if $\varepsilon, \psi(\cdot) > 1$.[24] Assumption 2 is trivially satisfied.

We henceforth focus on the case in which $\psi(\cdot) = \varepsilon$ and denote the constant elasticity by ψ, with $u(c) = \frac{c^{1-1/\psi}}{1-1/\psi}$. This assumption significantly simplifies the analysis because it implies that a constant share, $\mu := \frac{1}{(b(1-1/\psi))^{\psi}+1}$, of production is consumed (with the rest exported). Additionally, we parameterize good 0 production function as $G(1-s) = \frac{1}{\beta}(1-s)^{\beta}$ with $\beta \in (0,1)$.

The locator function. As shown in Appendix D.1, the locator function is given by

$$s \mapsto \mathcal{L}(s,\delta) = (\alpha + \delta\theta)\mu^{-\frac{1}{\psi}} s^{(\theta+\alpha)(1-1/\psi)-1} - (1-s)^{\beta-1}.$$

Without learning by doing this would be $\alpha\mu^{-\frac{1}{\psi}} s^{\alpha(1-1/\psi)-1} - (1-s)^{\beta-1}$. The first term is the marginal utility from allocating an additional unit of labor to sector 1, while the second term is the corresponding opportunity cost in terms of good 0. The presence of learning by doing implies an increase in the first term associated with the discounted gain from higher knowledge next period. In addition, learning partially offsets the curvature associated with decreasing returns, as captured by $\theta + \alpha$ in the exponent for s.

Low learning elasticity. If $\theta < \frac{1}{\psi-1} + 1 - \alpha$, the learning elasticity is low relative to the curvature associated with decreasing marginal utility and decreasing returns (captured by $\frac{1}{\psi-1}$ and $1-\alpha$), then all optimal paths converge to a unique, interior, steady state.[25]

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[24] Note that $\psi(\cdot) > 1$ is not a necessary condition. If exports are high relative to consumption and if foreign demand is highly elastic then most of the extra output associated with a higher s' will be exported, so we can have $\pi_{12}(s,s') > 0$ even with $\psi(\cdot)$ lower than 1.

[25] The condition on the learning elasticity can be seen most easily in the case of a static, closed economy in which the planner chooses employment in sector 1 to maximize utility, i.e., $\max_x \left(x^{\theta}x^{\alpha}\right)^{1-1/\psi} + 1 - x$. In this case, the condition for an interior solution is precisely $\theta < \frac{1}{\psi-1} + 1 - \alpha$.

Proposition 5.1. *If $\theta < \frac{1}{\psi-1} + 1 - \alpha$ then there exists a unique, globally stable in the interior, steady state.*

Proof. See Appendix D.1. □

Thus, if the learning elasticity θ is sufficiently low, the economy converges to the same level of labor regardless of the initial level of labor in sector 1 as long as it is not at a corner.

High learning elasticity. The dynamics are very different if the learning elasticity is high, $\theta > \frac{1}{\psi-1} + 1 - \alpha$. Given our parametric assumptions, $s \mapsto \mathcal{L}(s, \delta)$ is concave (see Lemma D.1 in Appendix D.1). Since $\mathcal{L}(0, \delta) = -1$ and $\lim_{s \to 1} \mathcal{L}(s, \delta) = -\infty$ then there are at most two interior steady states. If $\mathcal{L}(s, \delta) < 0$ for all $s \in [0, 1]$ then there are no roots and thus no interior steady states. By Lemma C.1 in Appendix C, $s = 1$ cannot be a stable steady state, and hence $s = 0$ is the unique, globally stable steady state. This corresponds to the case in which neither domestic nor foreign demand is sufficient to justify the opportunity cost of allocating labor to the production of good 1.

A more interesting case arises when there are two interior roots, which we denote by $s_* < s^*$, as in Proposition 4.2. Since $s' \mapsto \pi(s, s')$ is strictly concave (given $\alpha, \beta \in (0, 1)$), we can invoke the last part of that proposition to conclude that, if $\mathcal{L}(s, 0) = \pi_2(s, s) = 0$ admits an interior solution, then s^* is a steady state and is locally stable. The function $\mathcal{L}(s, 0)$ is inverted-U shaped and is negative at $s = 0$ and $s = 1$. Moreover, $\mu^{-1/\psi}$ is increasing and unbounded in the foreign demand shifter b. It follows that there exists a threshold $\bar{b}$ such that $\mathcal{L}(s, 0)$ has a unique root in $s \in [0, 1]$ when $b = \bar{b}$. For $b > \bar{b}$, the function $\mathcal{L}(s, 0)$ has two interior roots; we denote the lower one by $\bar{s}_0$.

Proposition 5.2. *Assume that $\theta > \frac{1}{\psi-1} + 1 - \alpha$, $\beta \in (0, 1)$ and $b > \bar{b}$, then s^* (the highest root of the locator function) is the unique locally stable interior steady state and the lower bound of its basin of attraction is below $\bar{s}_0$.*

Proof. $\bar{b}$ is defined at (7), and the proof is in Appendix D.1. □

While our method does not characterize the exact basin of attraction of stable steady states, it provides a conservative estimate of its left boundary by studying the dynamics of the myopic problem, a considerably simpler problem that can be solved with "pen-and-paper." This yields useful insights into optimal paths in our economy with learning externalities. If the learning elasticity is low, there is a unique globally stable interior steady state. If instead the learning elasticity is high and the learning sector faces sufficiently strong foreign demand, the lower root of the myopic planner's locator function gives an upper bound on the threshold above which the economy converges to a steady state with a positive learning sector.

6 Remarks

In this section, we review existing approaches for analyzing steady states and their stability, discuss how our proposed method relates to them, and conclude with a discussion of the role played by Assumption 1.

Taxonomy of the existing literature. While the stability of steady states can always be analyzed on a case-by-case basis, the general analytical approaches used in the literature fall roughly into two categories: the Lyapunov method and linear approximation methods. The Lyapunov approach (e.g., Brock and Scheinkman (1976); Stokey et al. (1989), Section 6.2) is global in nature but there is almost no guidance on how to establish existence or construct the Lyapunov function, thereby limiting its applicability in many settings.[26]

Linear approximation methods require additional assumptions, such as strict concavity of the per-period payoff, and are both local in nature and uninformative about the precise shape of the basin of attraction.[27] To fix ideas, we consider a version of the approach presented in Sections 6.2 and 6.3 of Stokey et al. (1989), and summarized in Theorem 6.9.

We formally discuss this theorem and its connection to our results below, but the main takeaway is that Theorem 6.9 assumes both strict concavity of the per-period payoff and uniqueness of the steady state—assumptions not required by our method. More importantly, a key limitation of this result is that it only guarantees the existence of a local neighborhood around the steady state in which convergence occurs, it does not characterize this neighborhood. As a result, given an initial condition, one cannot determine whether it lies within this region, and thus whether convergence to the steady state will

[26] See the discussion in Stokey et al. (1989) p. 139-140.

[27] To our knowledge, global results based on linear approximations are quite limited, and have been obtained only for the Cass–Koopmans growth model; see Nishimura et al. (2004) and references therein.

in fact occur. We also point out how, for cases in which π is *not* strictly concave the aforementioned method cannot be implemented. Our method, on the other hand, can be applied, albeit delivering conservative estimates of the basin of attractions.

Relationship with local approximation results. We now present a more thorough discussion of the similarities and differences between Proposition 4.3 and the local approximation method, represented by Theorem 6.9 in Stokey et al. (1989). The approach in Stokey et al. (1989) considers the dynamical system $\zeta_{t+1} = \Xi(\zeta_t) := (\zeta_t^{(2)}, G(\zeta_t))$, where $G : \mathbb{S}^2 \to \mathbb{S}$ is defined implicitly by $0 = \pi_2(\zeta^{(1)}, \zeta^{(2)}) + \delta\pi_1(\zeta^{(2)}, G(\zeta))$ for any $\zeta = (\zeta^{(1)}, \zeta^{(2)}) \in \mathbb{S}^2$. It is clear that a fixed point of Ξ will be a fixed point of $\Gamma(\cdot, \delta)$. Moreover there exists an equivalence among the respective flows: A flow of Ξ, $(\zeta_t)_t$ can be seen as $\zeta_t = (s_t, s_{t+1})$ where $(s_t)_t$ is a flow of $\Gamma(\cdot, \delta)$. Hence, it suffices to analyze the stability properties of fixed points Ξ.

Stokey et al. (1989) study the Jacobian of Ξ at a fixed point $\zeta = (s, s)$,

$$J_\Xi(\zeta) = \begin{bmatrix} 0 & 1 \\ G_1(\zeta) & G_2(\zeta) \end{bmatrix},$$

and claim that local stability of a fixed point ζ suffices to show J_Ξ has a unique characteristic root with absolute value less than one (cf. Theorem 6.9). J_Ξ's characteristic polynomial is given by $\lambda^2 - \lambda G_2(\zeta) - G_1(\zeta) = 0$. By the IFT the derivatives of G are given by $G_1(\zeta) = -\pi_{12}(\zeta^{(1)}, \zeta^{(2)})/(\delta\pi_{12}(\zeta^{(2)}, G(\zeta)))$ and $G_2(\zeta) = -(\pi_{22}(\zeta^{(1)}, \zeta^{(2)}) + \delta\pi_{11}(\zeta^{(2)}, G(\zeta)))/(\delta\pi_{12}(\zeta^{(2)}, G(\zeta)))$ respectively. Evaluating these expressions at $\zeta = (s, s)$ and some straightforward algebra implies the following characterization of the characteristic polynomial: $\lambda^2 + \lambda(\pi_{22}(s, s) + \delta\pi_{11}(s, s))/(\delta\pi_{12}(s, s)) + 1/\delta = 0$. Since the LHS is a quadratic function and coefficient of λ is negative, by Vieta's formula the two roots should both be positive. Then note that LHS at $\lambda = 0$ is positive, so existence of a positive root with absolute value less than one is implied by the LHS being negative at $\lambda = 1$; i.e., $\delta\pi_{12}(s, s) + \pi_{22}(s, s) + \delta\pi_{11}(s, s) + \pi_{12}(s, s) < 0$. This is *exactly* the condition of $\mathcal{L}_1(s, \delta) < 0$ which we derived in Theorem 1. That is, the linear approximation method uses the *same* mathematical quantity–the sign of $\mathcal{L}_1(s, \delta)$– to assess whether a fixed point is stable. The difference, however, lies in how each approach arrives at this result.

Instead of using linear approximations (and strict concavity of π), our approach relies on the locator function, which is linked to the dynamical behavior under "order conditions" (Assumption 1), and on drawing insights from the Samuelson principle and comparative statics. This discrepancy has important implications in terms of the characterization of the basins of attraction. Theorem 6.9 in Stokey et al. (1989) only proves an open neighborhood of initial conditions leading to convergence but does not specify it, limiting its applicability since one cannot determine if a given initial condition will converge. Proposition 4.3 on the other hand, provides, under an easy-to-check additional condition, a complete characterization of the basin of attractions and of steady states.

On the role of Assumption 1. As noted above, our approach replaces the strict concavity assumption with the monotonicity conditions outlined in Assumption 1. Here we discuss their role and whether they can be relaxed.

Our results extend to the case in which Assumption 1(i) is replaced by $\pi_1 < 0$. What matters for our analysis is therefore not the *positive* sign of π_1 per se, but rather that it have a *definite* sign.[28] By contrast, Assumption 1(iii) — namely, $\pi_{12} > 0$, which implies that payoffs exhibit increasing differences between the current and future state — appears to be essential for our results. We now provide an heuristic explanation. [29]

The key result underpinning Theorem 1 is Lemma 3.4, a comparative statics result establishing monotonicity of $\delta \mapsto \Gamma(s, \delta)$. Intuitively, comparative statics of the optimal policy correspondence Γ with respect to the discount factor are governed by two forces: a *direct effect*, capturing how a higher discount factor changes the weight placed on the continuation value in the overall payoff, and an *indirect effect*, capturing how a higher discount factor shifts the continuation value itself through changes in future states. The condition on the sign of π_1 is used solely to ensure that the value function is monotone in the state — decreasing if $\pi_1 < 0$ and increasing if $\pi_1 > 0$ — which in turn guarantees that the direct effect of a higher discount factor is monotone in the state.

The indirect effect, on the other hand, is governed jointly by the monotonicity of the value function and of the optimal policy correspondence. Crucially, the optimal policy

[28] For ease of exposition, we abstract from potential non-differentiability and assume that π is twice continuously differentiable.

[29] For a more detailed and formal treatment, we refer the reader to Supplemental Material SM.8.

correspondence has to be *increasing* (as a function of s), and this is ensured by $\pi_{12} > 0$, regardless of the sign of π_1. As a result, as long as $\pi_{12} > 0$, the direct and indirect effects reinforce each other, yielding a positive total effect when $\pi_1 > 0$ and a negative one when $\pi_1 < 0$. Consequently, when $\pi_1 < 0$ (and $\pi_{12} > 0$), the conclusion of Theorem 1 continues to hold: for any interior steady state s, we have $s \in \mathcal{R}^o[\mathcal{L}]$, with s stable if $\mathcal{L}_1(s, \delta) < 0$ and unstable if $\mathcal{L}_1(s, \delta) > 0$.[30]

As an illustration, we consider a simple model of rational addiction (Becker and Murphy, 1988). An agent with unit income allocates $x \in [0, 1]$ to an addictive good (price normalized to one), with addiction summarized by a state variable s evolving as $s' = (1 - d)s + x$. Per-period utility is $u(s, x) = -s^{\alpha} - bs^{\beta}(1 - x)^2 + c(1 - x)$, with $b, c > 0$ and $\alpha, \beta \in (0, 1)$. The term $-s^{\alpha}$ captures long-run harms from addiction, $-bs^{\beta}(1 - x)^2$ generates reinforcement (i.e., higher s raises the marginal cost of abstention), and $c(1 - x)$ captures instantaneous utility from consuming non-addictive goods. Defining the state space as $\mathbb{S} = [0, 1/d]$, we have $\pi_1(s, s') < 0$ for all feasible (s, s') if $\alpha d^{1-\alpha} > c(1 - d)$ (which we assume henceforth), while $\pi_{12} > 0$ follows directly. Simple algebra shows that $\lim_{s \to 0} \mathcal{L}(s, \delta) = -\infty$ while $\mathcal{L}(1/d, \delta) < 0$, hence either the locator function has no roots or it has two roots. If it has no roots, then $s = 0$ is a globally stable steady state, while if it has two roots then we know from Theorem 1 that the higher root is the only possible interior locally stable steady state. Thus, in this case, the higher root identifies the potential addictive steady state, and we know from Proposition 4.2 that the only other possibility is that $s = 0$ is the global stable steady state.

We conclude this section by pointing out that having a definite sign on π_1 is indeed necessary — in the sense that without it, we can find an example that violates the conclusion of Theorem 1.[31] Consider the problem where $\pi(s, s') := -0.5(s' - g(s))^2$ with $g : \mathbb{S} \to \mathbb{S}$ increasing and smooth and with only one interior fixed point, a, that satisfies $g'(a)\delta > 1$. This fixed point is clearly unstable. Observe that $\pi_1(s, s') = g'(s)(s' - g(s))$ does not have a definite sign, in particular around a. It is easy to see that the policy correspondence is a function equal to g, so a is an interior, unstable, steady state. However,

[30] Going back to Equation 3, the LHS reverses sign since now an increase in δ decreases rather than increases the steady state s, however this is balanced by the sign reversal of π_1 and hence $\mathcal{L}_1(q(\delta), \delta)$ does not change sign.

[31] We are grateful to Ivan Werning for posing this question to us.

the locator function is given by $s \mapsto -(s - g(s))(1 - \delta g'(s))$, which at the point a has a derivative given by $-(1 - g'(a))(1 - \delta g'(a))$ which is negative because $1 < 1/\delta < g'(a)$. This violates Theorem 1(2).

State Space. The state space, $\mathbb{S}$, is assumed to be a bounded, convex, one-dimensional set; that is, an interval. Convexity is a technical convenience that permits differentiation of the relevant functions. Boundedness can be relaxed under restrictions on the tail behavior of π, as shown in Supplemental Material SM.1. As discussed there, an unbounded state space allows some optimal paths to drift to plus/minus infinity, but does not otherwise affect our results.

The assumption of a one-dimensional state space is central to tractability: it makes the state space totally ordered and allows steady states and their local stability to be characterized using scalar derivatives. In higher dimensions, extending Theorem 1 and its refinements is in principle possible under suitable regularity and monotonicity conditions on the gradient and cross-derivative matrix of the payoff function. We view this as a fruitful open question, since it would substantially broaden the scope of the theory. However, such an extension introduces significant technical complications and is beyond the scope of this paper. Moreover, in higher dimensions, it is no longer immediate that the asymptotic behavior of optimal paths is fully described by fixed points, as in Proposition 3.2; cycles or more complex invariant sets may not be ruled out *a priori*.

7 Conclusion

We have introduced a monotonicity-based approach for characterizing optimal paths in dynamic optimization problems. By centering the analysis on a simple locator function constructed from model primitives, we can identify steady states, assess their stability, and in some cases describe their basins of attraction, all without solving the full dynamic program. Applications to a generalized neoclassical growth model and an economy with learning-by-doing illustrate both the tractability and the breadth of the framework, and point to its usefulness for applied work that seeks qualitative insights.

The main limitation of the present analysis is its restriction to a one-dimensional state space. Extending the framework to multiple states, where cycles, richer path dependence, and interactions across dimensions may arise, remains an important avenue for future research. Developing higher-dimensional analogues of the locator function would significantly expand the scope of monotonicity-based methods for analyzing optimal paths.

References

Aiyagari, S. R. (1994). Uninsured idiosyncratic risk and aggregate saving. *The Quarterly Journal of Economics*, 109(3):659–684.

Alvarez, F. and Stokey, N. L. (1998). Dynamic programming with homogeneous functions. *Journal of Economic Theory*, 82(1):167–189.

Azariadis, C. and Drazen, A. (1990). Threshold externalities in economic development. *The Quarterly Journal of Economics*, 105(2):501–526.

Bardhan, P. K. (1971). On optimum subsidy to a learning industry: An aspect of the theory of infant-industry protection. *International Economic Review*, 12(1):54–70.

Becker, G. S. and Murphy, K. M. (1988). A theory of rational addiction. *Journal of Political Economy*, 96(4):675–700.

Ben-Porath, Y. (1967). The production of human capital and the life cycle of earnings. *Journal of Political Economy*, 75(4, Part 1):352–365.

Bewley, T. F. (1986). Stationary monetary equilibrium with a continuum of independently fluctuating consumers. *In Werner Hildenbrand and Andreu Mas-Colell (eds.), Contributions to Mathematical Economics in Honor of Gerard Debreu*, pages 79–102.

Brock, W. A. and Scheinkman, J. A. (1976). Global asymptotic stability of optimal control systems with applications to the theory of economic growth. In *The Hamiltonian approach to dynamic economics*, pages 164–190. Elsevier.

Clark, C. W. (1973). Profit maximization and the extinction of animal species. *Journal of Political Economy*, 81(4):950–961.

Clausen, A. and Strub, C. (2016). A general and intuitive envelope theorem. Discussion Paper 274, Edinburgh School of Economics, University of Edinburgh.

Cotter, K. D. and Park, J.-H. (2006). Non-concave dynamic programming. *Economics Letters*, 90(1):141–146.

Debreu, G. (1970). Economies with a finite set of equilibria. *Econometrica*, 38(3):387–392.

Dechert, W. D. and Nishimura, K. (2012). A complete characterization of optimal growth paths in an aggregated model with a non-concave production function. *Nonlinear Dynamics in Equilibrium Models: Chaos, Cycles and Indeterminacy*, pages 237–257.

Durlauf, S. N. and Johnson, P. A. (1995). Multiple regimes and cross-country growth behaviour. *Journal of Applied Econometrics*, 10(4):365–384.

Echenique, F. (2000). Comparative statics by adaptive dynamics and the correspondence principle. Economics Working Papers E00-273, University of California at Berkeley.

Echenique, F. (2002). Comparative statics by adaptive dynamics and the correspondence principle. *Econometrica*, 70(2):833–844.

Grüne, L., Semmler, W., and Sieveking, M. (2005). Creditworthiness and thresholds in a credit market model with multiple equilibria. *Economic Theory*, 25(2):287–315.

Hayashi, F. (1982). Tobin's marginal q and average q: A neoclassical interpretation. *Econometrica*, pages 213–224.

Huggett, M. (1993). The risk-free rate in heterogeneous-agent incomplete-insurance economies. *Journal of Economic Dynamics and Control*, 17(5-6):953–969.

Kamihigashi, T. and Roy, S. (2007). A nonsmooth, nonconvex model of optimal growth. *Journal of Economic Theory*, 132(1):435–460.

Kiyotaki, N. and Moore, J. (1997). Credit cycles. *Journal of Political Economy*, 105(2):211–248.

Krugman, P. (1987). The narrow moving band, the dutch disease, and the competitive consequences of mrs. thatcher: Notes on trade in the presence of dynamic scale economies. *Journal of Development Economics*, 27(1-2):41–55.

Lucas Jr, R. E. (1988). On the mechanics of economic development. *Journal of Monetary Economics*, 22(1):3–42.

Majumdar, M. and Mitra, T. (1982). Intertemporal allocation with a non-convex technology: the aggregative framework. *Journal of Economic Theory*, 27(1):101–136.

Melitz, M. J. (2005). When and how should infant industries be protected? *Journal of International Economics*, 66(1):177–196.

Milgrom, P. and Shannon, C. (1994). Monotone comparative statics. *Econometrica*, pages 157–180.

Nishimura, K., Stachurski, J., et al. (2004). Discrete time models in economic theory. *Cubo–A Mathematical Journal*, 6:187–207.

Redding, S. J. (1999). Dynamic comparative advantage and the welfare effects of trade. *Oxford Economic Papers*, 51(1):15–39.

Samuelson, P. A. (1983). *Foundations of economic analysis*. Harvard university press.

Skiba, A. K. (1978). Optimal growth with a convex-concave production function. *Econometrica*, pages 527–539.

Stokey, N. L., Lucas Jr, R. E., and Prescott, E. C. (1989). *Recursive methods in economic dynamics*. Harvard University Press.

Young, A. (1991). Learning by doing and the dynamic effects of international trade. *The Quarterly Journal of Economics*, 106(2):369–405.

Appendix

A Appendix for Section 3.1

Proof of Lemma 3.1. $(\delta, s) \mapsto V(s,\delta)$ is unique, continuous and increasing by Lemmas SM.2.1 and SM.2.2 in Supplemental Material SM.2. $\square$

Proof of Lemma 3.2. $(s,\delta) \mapsto \Gamma(s,\delta)$ is non-empty, compact-valued, and upper hemicontinuous, also $s \mapsto \Gamma(s,\delta)$ is function-like and non-decreasing by Lemmas SM.3.1 and SM.3.3 in Supplemental Material SM.3. $\square$

Proof of Proposition 3.1. We first claim that there exists a $\Delta > 0$ small enough such that there exists a $y_\Delta \in \Gamma(s+\Delta,\delta)$ such that $y_\Delta \in Y(s)$. By UHC of Γ (Lemma 3.2), and $\Gamma(s,\delta) \subseteq Y^o(s)$, we know there exists Δ such that $\Gamma(s+\Delta,\delta) \subseteq Y^o(s)$. That is because for any given $\Delta_n \to 0$, any sequence of $\Gamma(s+\Delta_n,\delta)$ will converge to an element of $\Gamma(s,\delta)$, which is in the interior of $Y^o(s)$.

From this claim it follows that $\frac{V(s+\Delta,\delta)-V(s,\delta)}{\Delta} \leq \frac{\pi(s+\Delta,y_\Delta)-\pi(s,y_\Delta)}{\Delta}$. Since $s \mapsto \Gamma(s,\delta)$ is UHC (Lemma 3.2), by possibly going to a subsequence, it follows that $y = \lim_{\Delta\to 0} y_\Delta \in \Gamma(s,\delta)$. By Assumption 1, $\lim_{\Delta\to 0} \frac{\pi(s+\Delta,y_\Delta)-\pi(s,y_\Delta)}{\Delta} = \partial_1^+\pi(s,y) \leq \max_{y\in\Gamma(s,\delta)} \partial_1^+\pi(s,y)$. Thus, $\limsup_{\Delta\downarrow 0} \frac{V(s+\Delta,\delta)-V(s,\delta)}{\Delta} \leq \max_{y\in\Gamma(s,\delta)} \partial_1^+\pi(s,y)$.

We now claim that there exists a $\Delta > 0$ small enough such that $\Gamma(s,\delta) \subseteq Y(s+\Delta)$. This follows by the fact that $\Gamma(s,\delta) \subseteq Y^o(s)$ and continuity of Y (Assumption 2). Hence, for all $y \in \Gamma(s,\delta)$,$\frac{V(s+\Delta,\delta)-V(s,\delta)}{\Delta} \geq \frac{\pi(s+\Delta,y)-\pi(s,y)}{\Delta}$. Thus, $\liminf_{\Delta\downarrow 0} \frac{V(s+\Delta,\delta)-V(s,\delta)}{\Delta} \geq \partial_1^+\pi(s,y)$. Since this holds for all $y \in \Gamma(s,\delta)$, it holds for the maximal element, and thus we showed:

$$\partial^+ V(s,\delta) = \lim_{\Delta\downarrow 0} \frac{V(s+\Delta,\delta)-V(s,\delta)}{\Delta} = \max_{y\in\Gamma(s,\delta)} \partial_1^+\pi(s,y).$$

We now analyze $\partial^- V(s,\delta)$. We first claim that there exists a $\Delta < 0$ small enough such that there exists a $y_\Delta \in \Gamma(s+\Delta,\delta)$ such that $y_\Delta \in Y(s)$. By UHC of Γ (Lemma 3.2), and $\Gamma(s,\delta) \subseteq Y^o(s)$, this argument is correct by the same argument for $\Delta > 0$ in the beginning.

Hence, for any $\Delta < 0$ and $y_\Delta \in \Gamma(s+\Delta,\delta) \cap Y(s)$, $\frac{V(s+\Delta,\delta)-V(s,\delta)}{\Delta} \geq \frac{\pi(s+\Delta,y_\Delta)-\pi(s,y_\Delta)}{\Delta}$. By taking limits it follows that for some $y \in \Gamma(s,\delta)$, $\liminf_{\Delta\uparrow 0} \frac{V(s+\Delta,\delta)-V(s,\delta)}{\Delta} \geq \partial_1^-\pi(s,y) \geq$

$\min_{y\in\Gamma(s,\delta)} \partial_1^- \pi(s,y)$. Analogous calculations to those above imply $\limsup_{\Delta\uparrow 0} \frac{V(s+\Delta,\delta)-V(s,\delta)}{\Delta} \leq \partial_1^- \pi(s,y)$ for all $y \in \Gamma(s,\delta)$. And thus,

$$\partial^- V(s,\delta) = \lim_{\Delta\uparrow 0} \frac{V(s+\Delta,\delta)-V(s,\delta)}{\Delta} = \min_{y\in\Gamma(s,\delta)} \partial_1^- \pi(s,y).$$

□

Proof of Corollary 3.1. Lemma SM.2.5 in Supplemental Material SM.2. □

B Appendix for Section 3.2

We prove a more general version of Proposition 3.2 under unbounded state space — it is obvious this proposition implies Proposition 3.2.

Proposition B.1. *For any $s_0 \in \mathbb{S}$ and any $\phi(.,s_0) \in \Phi(s_0)$, $\lim_{t\to\infty} \phi(t,s_0) \in \mathcal{R}[\bar{\Gamma}] \cup \{\pm\infty\}$.*

Proof of Proposition B.1. For any s_0 either $\phi_\delta(1,s_0) > s_0$, $\phi_\delta(1,s_0) < s_0$ or $\phi_\delta(1,s_0) = s_0$. In the third case the result trivially holds so we focus on the other two. Suppose $\phi_\delta(1,s_0) < s_0$. Since $\Gamma(\cdot,\delta)$ is non-decreasing (see Lemma 3.2) it follows that $\Gamma(\phi_\delta(1,s_0),\delta) \leq \Gamma(s_0,\delta)$ which implies that $\phi_\delta(2,s_0) \leq \phi_\delta(1,s_0) < s_0$. By applying this logic recursively we obtain that $(\phi_\delta(t,s_0))_t$ is a non-increasing sequence. Hence, the limit exists and is given by $r = \inf_t \phi_\delta(t,s_0)$. Observe that r is either finite or infinite, if it is finite, since $\Gamma(\cdot,\delta)$ is UHC (see Lemma 3.2), $r \in \Gamma(r,\delta)$. By Lemma SM.3.2 in the Supplemental Material SM.3 this implies that $r \in \mathcal{R}[\bar{\Gamma}]$ as desired.

Suppose $\phi_\delta(1,s_0) > s_0$. By analogous arguments to those above, $(\phi_\delta(t,s_0))_t$ is a non-decreasing sequence, hence the limit exists and is given by $s := \sup_t \phi_\delta(t,s_0)$. Observe that s is either finite or infinite, if it is finite, since $\Gamma(\cdot,\delta)$ is UHC (see Lemma 3.2), $s \in \Gamma(s,\delta)$. By Lemma SM.3.2 this implies that $s \in \mathcal{R}[\bar{\Gamma}]$ as desired. □

Remark B.1. The result allows for $\mp\infty$ to be a limit point. This is natural as in this proposition $\mathbb{S}$ can be unbounded, so some paths of Γ may drift to infinity. A sufficient condition to rule out these case is $\limsup_{s\to\infty} \max \mathrm{Y}(s)/s < 1$ and $\liminf_{s\to-\infty} \min \mathrm{Y}(s)/s > 1$. △

C Appendix for Section 4

The proof of the propositions in Section 4 rely on the following lemma which provides information for the behavior of the optimal policy correspondence at the boundary.

Lemma C.1. *Suppose $\mathbb{S} = [\underline{s}, \bar{s}]$ and $s \in \Upsilon^o(s)$ for all $s \in \mathbb{S}^o$, and suppose π_1 and π_2 are continuous. The following are true:*

1. *If $\mathcal{L}(\underline{s}, \delta) > 0$ then either $\Gamma(\underline{s}, \delta) > \underline{s}$ or $\exists\, \varepsilon > 0$ such that $\Gamma(s, \delta) > s,\ \forall s \in (\underline{s}, \underline{s} + \varepsilon)$.*

2. *If $\mathcal{L}(\bar{s}, \delta) < 0$ then either $\Gamma(\bar{s}, \delta) < \bar{s}$ or $\exists\, \varepsilon > 0$ such that $\Gamma(s, \delta) < s,\ \forall s \in (\bar{s} - \varepsilon, \bar{s})$.*

3. *If $\mathcal{L}(\underline{s}, \delta) < 0$ and $\Gamma(\underline{s}, \delta) = \underline{s}$, then $\exists\, \varepsilon > 0$ such that $\Gamma(s, \delta) < s,\ \forall s \in (\underline{s}, \underline{s} + \varepsilon)$.*

4. *If $\mathcal{L}(\bar{s}, \delta) > 0$ and $\Gamma(\bar{s}, \delta) = \bar{s}$, then $\exists \varepsilon > 0$ such that $\Gamma(s, \delta) > s,\ \forall s \in (\bar{s} - \varepsilon, \bar{s})$.*

Proof. See Supplemental Material SM.5 □

C.1 Proof of Proposition 4.1

Lemma C.2. *Suppose $\mathbb{S} = [\underline{s}, \bar{s}]$ and $s \in \Upsilon^o(s)$ for all $s \in \mathbb{S}^o$, and suppose π_1 and π_2 are continuous. If $\mathcal{R}[\mathcal{L}] = \{e\}$, then the following are true:*

1. *If $\mathcal{L}(\underline{s}, \delta) > 0$, then there exists a $c \in \mathbb{S}$ such that $(\underline{s}, c) \subseteq \mathcal{B}(e, \delta)$ and $(c, \bar{s}] \subseteq \mathcal{B}(\bar{s}, \delta)$.*

2. *If $\mathcal{L}(\bar{s}, \delta) < 0$, then there exists a $c \in \mathbb{S}$ such that $[\underline{s}, c) \subseteq \mathcal{B}(\underline{s}, \delta)$ and $(c, \bar{s}) \subseteq \mathcal{B}(e, \delta)$.*

Proof. See Supplemental Material SM.5. □

Proof of Proposition 4.1. Under our assumptions, the conditions for Lemma C.2 hold. Moreover, since the locator function satisfies single-crossing from above, both parts of Lemma C.2 hold. Observe that c in Part 1, which we denote as c_1, must equal $\bar{s}$. Otherwise, it would contradict the claim about $\mathcal{B}(e, \delta)$ in part 2. Similarly, the c in Part 2, which we denote as c_2, equals $\underline{s}$; otherwise it would contradict part 1.

This shows that the interior unique root of $s \mapsto \mathcal{L}(s, \delta)$ is globally stable over $\mathbb{S}^o$. We now show that such root must be a steady state. To show this, observe that by the proof of Lemma C.2, $\Gamma(s, \delta) > s$ for some s arbitrarily close to $\underline{s}$ and $\Gamma(s, \delta) < s$ for some s arbitrarily close to $\bar{s}$. Since $\Gamma(\cdot, \delta)$ is non-decreasing, these inequalities imply that

$s \mapsto \Gamma(s,\delta) - s$ must cross zero. By Lemma SM.3.2, it must do so at a fixed point and by Theorem 1, it thus must be a root. Thus the desired result holds. □

C.2 Proof of Proposition 4.2

Proof of Proposition 4.2. Since $\mathcal{L}(\bar{s},\delta) < 0$, by Lemma C.1(2), $\Gamma(s,\delta) < s$ for all s in an open ball to the left of $\bar{s}$. Therefore, one possibility is $\Gamma(s,\delta) < s$ for all $s \in \mathbb{S}^o$. In this case no interior steady states exist and $\underline{s}$ is the only (globally stable) steady state.

Another possibility, however, is for $\Gamma(\cdot,\delta)$ to intersect the 45° line for the first time (coming "from the right") in some interior point, denoted by e. By Theorem 1 e must be one of the roots. But since $s \mapsto \bar{\Gamma}(s,\delta)$ is decreasing at such point, e is a stable steady state.Thus, by Theorem 1, $e = s^*$.

From s^* and moving "to the left", we claim $\Gamma(\cdot,\delta)$ can intersect the 45° at most one time over $(\underline{s}, s^*)$. This claim follows directly from Theorem 1 and the assumption that the locator function has only two roots. Also by the same theorem, we can conclude that the only possible fixed point over $(\underline{s}, s^*)$ would be located at s_* and would be unstable. The other two possibilities are for $\Gamma(\cdot,\delta)$ to "jump" below the 45° line and remain below the 45° line until we reach $\underline{s}$ or to remain above the 45° line until we reach $\underline{s}$.

We have shown that if the locator function has two roots and $\mathcal{L}(\underline{s},\delta), \mathcal{L}(\bar{s},\delta) < 0$, then either $\underline{s}$ is a globally stable steady state, or s^* is the only interior (locally) stable one.

Now suppose there exists $s_0 \in \mathbb{S}^o$ such that $\pi_2(s_0,s_0) = 0$. Since $s' \mapsto \pi(s_0,s')$ is strictly concave, this implies that $\Gamma(s_0,0) = s_0$. By Lemma 3.2, $\Gamma(s_0,\delta) \geq \Gamma(s_0,0) = s_0$, and hence $\Gamma(s,\delta) < s$, $\forall s \in \mathbb{S}^o$ cannot hold. So case one in the lemma is ruled out. □

C.3 Proof of Proposition 4.3

To establish Proposition 4.3 we employ the following lemmas.

Lemma C.3. *Suppose π is strictly concave, $s \in \Upsilon(s)$ for all $s \in \mathbb{S}^o$, and for all $s \in \mathcal{R}^o[\mathcal{L}]$,*

$$\max_{a \in \Upsilon(s):\, a > s} \pi_2(s,a) + \delta\pi_1(s,a) \leq 0 \leq \min_{a \in \Upsilon(s):\, a < s} \pi_2(s,a) + \delta\pi_1(s,a). \tag{5}$$

Then s is an interior steady state if and only if it is an interior root of the locator function.

Proof. See Supplemental Material SM.5. $\square$

Lemma C.4. *Suppose $s \mapsto \Gamma(s,\delta)$ is a function. Then, for all interior steady state e, $\mathcal{B}(e,\delta) \supseteq \mathcal{B}[\mathcal{L}](e)$.*

Proof. See Supplemental Material SM.5. $\square$

Proof of Proposition 4.3. Since π is strictly concave, so is $V(\cdot,\delta)$ and thus $\Gamma(\cdot,\delta)$ is a function. Thus, by Lemma C.4, $\mathcal{B}[\mathcal{L}](s) \subseteq \mathcal{B}(s,\delta)$ for any interior steady state s.

We now show $\mathcal{B}[\mathcal{L}](s) \supseteq \mathcal{B}(s,\delta)$ for any interior steady state s. If s is unstable this statement is trivially true, so we focus on s stable. Consider any $a \in \mathcal{B}(s,\delta)$ such that $a < s$.[32] Suppose $a \notin \mathcal{B}[\mathcal{L}(\cdot,\delta)](s)$. The only way $a \notin \mathcal{B}[\mathcal{L}(\cdot,\delta)](s)$ is that there exists a $b \in (a,s)$ that is a root of $\mathcal{L}(\cdot,\delta)$, but by Lemma C.3 b is a steady state, and Proposition 3.2, b is a fixed point of $\Gamma(\cdot,\delta)$, which implies that $b \notin \mathcal{B}(s,\delta)$. This yields a contradiction because $\mathcal{B}(s,\delta) = \mathcal{B}[\bar{\Gamma}(\cdot,\delta)](s)$ is convex. $\square$

D Appendix for Proof of Theorem 1

Proof of Lemma 3.3. For any $s \in \mathcal{R}^o[\bar{\Gamma}(\cdot,\delta)]$, $s \in \mathit{Range}(\Gamma(\cdot,\delta))$. Thus, optimality and Proposition 3.2 imply $0 = \pi_2(s,s) + \delta\pi_1(s,y)$ for any $y \in \Gamma(s,\delta)$. But since s is a fixed point, $y = s$. Hence, $\mathcal{L}(s,\delta) = 0$ and thereby the first part of Theorem 1 is proven. $\square$

Proof of Lemma 3.4. $\delta \mapsto \Gamma(s,\delta)$ non-decreasing by Lemma SM.3.4 in Supplemental Material SM.3 — the condition in the Lemma is satisfied by Lemma 3.1. $\square$

Proof of Lemma 3.5. We rely on Lemma 3.2 that implies that for any $s \in \mathbb{S}$ for which $\Gamma(s,\delta) \in \mathrm{Y}^o(s)$, and for any $\delta' > \delta$,[33]

$$\min \Gamma(s,\delta') \geq \max \Gamma(s,\delta). \tag{6}$$

We first show the case of stable steady state. We do this by contradiction, i.e., suppose there exists $(s,\delta) \in \mathit{Graph}\mathcal{E}^o$ with s stable such that for any open neighborhood of δ either

[32] The proof for the case $a > s$ is analogous and thus omitted.
[33] For a set S, $\min S := \min\{s \colon s \in S\}$ and analogously with $\max S$.

(a) there exists a $\delta' > \delta$ in this neighborhood such that $q(\delta') < q(\delta) = s$ or (b) there exists a $\delta' < \delta$ in this neighborhood such that $q(\delta') > q(\delta) = s$.

Suppose (a) holds (the proof for (b) is analogous and thus omitted). Since s is a stable steady state, by Proposition SM.3.1 in the Supplemental Material SM.3, there exists a $\underline{s} \in \mathbb{S}$ such that for any $s \in U(s) := (\underline{s}, s)$, $\Gamma(s, \delta) > s$. By expression 6 this implies that $\Gamma(s, \delta') \geq \Gamma(s, \delta) > s$ for any $\delta' > \delta$.

By our assumption and continuity of q, there exists a $\delta' > \delta$ such that $q(\delta') \in U(s)$. So, the previous display implies that $\Gamma(q(\delta'), \delta') > q(\delta')$, but this is a contradiction to the fact that $q(\delta') \in \mathcal{R}^o[\bar{\Gamma}(\cdot, \delta')]$. Hence, $q(\delta)$ is increasing in δ with $q(\delta)$ stable.

The proof for the case of unstable steady state is analogous. I.e., suppose there exists a $(u, \delta) \in Graph\mathcal{E}^o$ with u unstable, such that for any open neighborhood of δ either (a) there exists a $\delta' > \delta$ in this neighborhood such that $q(\delta') > q(\delta) = u$ or (b) there exists a $\delta' < \delta$ in this neighborhood such that $q(\delta') < q(\delta) = u$. Proposition SM.3.1 implies the existence of $\underline{u}, \overline{u} \in \mathbb{S}$ such that for any $s \in U_-(u) := (\underline{u}, u)$, $\Gamma(s, \delta) < s$ and for any $s \in U_+(u) := (u, \overline{u})$, $\Gamma(s, \delta) > s$. This result and expression 6 imply that for any $\delta' > \delta$, $\Gamma(s, \delta') > s$ for any $s \in U_+(u)$; and for any $\delta' < \delta$, $\Gamma(s, \delta') < s$ for any $s \in U_-(u)$. But these expressions contradict (a) and (b) respectively. $\square$

Remark D.1. Existence of $U_q(\delta)$ and the mapping q is ensured by our assumption of genericity. For non-generic fixed points, $q(\delta)$ exist, but there could exist a sequence $(\delta'_n)_n$ converging to δ for which either $q(\delta'_n)$ does not exist or $q(\delta'_n)$ is multi-valued (not a function). The first case could occur because $\bar{\Gamma}(\cdot, \delta'_n)$ does not intersect zero, the second case could occur because $\bar{\Gamma}(\cdot, \delta)$ does intersect zero but it does so tangentially. $\triangle$

D.1 Appendix for Section 5

Proof of Lemma 5.1. We show Assumption 1 holds by showing $\pi_1(s, s') > 0$ and $\pi_{12}(s, s') > 0$. By the Envelope Theorem, $\pi_1(s, s') = u'(c(s, s'))H'(s)F(s')$, which is positive. Also,

$$\begin{aligned}
\pi_{12}(s, s') =& u''(c(s, s'))\frac{\partial c(s, s')}{\partial s'}H'(s)F(s) + u'(c(s, s'))H'(s)F'(s') \\
=& u'(c(s, s'))H'(s)\left(F'(s') - \frac{1}{c(s, s')\psi(c)}\frac{\partial c(s, s')}{\partial s'}F(s')\right).
\end{aligned}$$

The first order condition determining c given s and s' can be written as $u'(c(s,s')) = \frac{\varepsilon-1}{\varepsilon}p(e(s,s'))$. In words, this condition states that the marginal utility of consuming good 1 must equal the marginal revenue of exporting this good, which is just the price divided by the markup $\varepsilon/(\varepsilon-1)$. Fix s and differentiate this condition, we have $u''(c(s,s'))dc(s,s') = \frac{\varepsilon-1}{\varepsilon}p'(H(s)F(s') - c(s,s'))(H(s)F'(s')ds' - dc(s,s'))$, which implies

$$\frac{\partial c(s,s')}{\partial s'} = \frac{\frac{\varepsilon-1}{\varepsilon}p'(e(s,s'))H(s)F'(s')}{\frac{\varepsilon-1}{\varepsilon}p'(e(s,s')) + u''(c(s,s'))} = \frac{\frac{\varepsilon-1}{\varepsilon}p'(e(s,s'))H(s)F'(s')}{\frac{\varepsilon-1}{\varepsilon}p'(e(s,s')) - \frac{(\varepsilon-1)p(e(s,s'))}{\varepsilon\psi(c(s,s'))c(s,s')}} > 0.$$

Plugging this back into our last expression of $\pi_{12}(s,s')$ yields, after some algebra,

$$\pi_{12}(s,s') = u'(c(s,s'))F'(s')H'(s)\frac{(\psi(c(s,s'))-1)\,c(s,s') + (\varepsilon-1)\,e(s,s')}{c(s,s') + (\varepsilon/\psi(c(s,s')))\,e(s,s')}.$$

Since $\varepsilon > 1$ and $\psi(\cdot) > 1$ then $(\psi(c(s,s'))-1)\,c(s,s') + (\varepsilon-1)\,e(s,s') > 0$, implying that $\pi_{12}(s,s') > 0$ and verifying Assumption 1. $\square$

Proposition D.1. *The locator function is:* $s \mapsto \mathcal{L}(s,\delta) = (\alpha+\delta\theta)\mu^{-\frac{1}{\psi}}s^{(\theta+\alpha)(1-1/\psi)-1} - G'(1-s)$.

Proof of Proposition D.1. By $\varepsilon = \psi$, $\pi(s,s') = \max_c \frac{c^{1-1/\psi}}{1-1/\psi} + b(s^\theta s'^\alpha - c)^{1-1/\psi} + G(1-s')$ the first order condition provides a constant consumption share $\mu = \frac{1}{(b(1-1/\psi))^\psi+1}$, and note that $\mu^{1-1/\psi} + b\frac{\psi-1}{\psi}(1-\mu)^{1-1/\psi} = \mu^{-\frac{1}{\psi}}$. The per-period payoff is

$$\begin{aligned}
\pi(s,s') =& \frac{\left(\mu s^\theta (s')^\alpha\right)^{1-1/\psi}}{1-1/\psi} + b\left((1-\mu)s^\theta(s')^\alpha\right)^{1-1/\psi} + G(1-s') \\
=& \left(\frac{\psi}{\psi-1}\mu^{1-1/\psi} + b(1-\mu)^{1-1/\psi}\right)\left(s^\theta(s')^\alpha\right)^{1-1/\psi} + G(1-s') \\
=& \frac{\psi}{\psi-1}\mu^{-\frac{1}{\psi}}\left(s^\theta(s')^\alpha\right)^{1-1/\psi} + G(1-s').
\end{aligned}$$

Taking derivatives, we have $\pi_1(s,s) = \theta\mu^{-\frac{1}{\psi}}s^{(\theta+\alpha)(1-1/\psi)-1}$, and $\pi_2(s,s) = \alpha\mu^{-\frac{1}{\psi}}s^{(\theta+\alpha)(1-1/\psi)-1} - G'(1-s)$. Therefore, $\mathcal{L}(s,\delta) = (\alpha+\delta\theta)\mu^{-\frac{1}{\psi}}s^{(\theta+\alpha)(1-1/\psi)-1} - G'(1-s)$. $\square$

Proof of Proposition 5.1. If $\theta < \frac{1}{\psi-1} + 1 - \alpha$, the locator function is decreasing by Proposition D.1. This follows because $\mathcal{L}_1(s,\delta) = (\alpha+\delta\theta)\mu^{-1/\psi}\left((\theta+\alpha)(1-1/\psi)-1\right)s^{(\theta+\alpha)(1-1/\psi)-2} +$

$G''(1-s)$. and since G is concave, $G'' < 0$. Moreover, by $\theta < \frac{1}{\psi-1} + 1 - \alpha$, $(\theta+\alpha)(1-1/\psi) - 1 < 0$. Thereby, the locator function is decreasing. In addition, $\lim_{s\downarrow 0} \mathcal{L}(s,\delta) = +\infty$, and $\mathcal{L}(1,\delta) = (\alpha+\delta\theta)\mu^{-1/\psi} - G'(0)$. Since $G'(0) = +\infty$, $\mathcal{L}(1,\delta) = (\alpha+\delta\theta)\mu^{-1/\psi} - G'(0) < 0$. Therefore, by Proposition 4.1 there is a unique root which is the unique, globally stable in the interior, steady state. □

Lemma D.1. *If $\theta > \frac{1}{\psi-1} + 1 - \alpha$, then $s \mapsto \mathcal{L}(s,\delta)$ is concave and $\mathcal{L}(0,\delta) = -G'(1) = -1$.*

Proof of Lemma D.1. The locator function is $\mathcal{L}(s,\delta) = (\alpha+\delta\theta)\mu^{-\frac{1}{\psi}} s^{(\theta+\alpha)(1-1/\psi)-1} - (1-s)^{\beta-1}$. Under our assumption of $\alpha, \theta \in (0,1)$, we have $(\theta+\alpha)(1-1/\psi) \in (1,2)$ and $\beta \in (0,1]$. Is easy to see that $s \mapsto s^{(\theta+\alpha)(1-1/\psi)-1}$ and $s \mapsto -(1-s)^{\beta-1}$ are concave. □

Let's define the critical foreign demand shifter $\bar{b}$ as follows.

$$\bar{b} := \frac{\psi}{\psi-1} \left\{ \left[\frac{\alpha}{(1-\beta)^{\beta-1}} \frac{((\theta+\alpha)(1-1/\psi)-1)^{(\theta+\alpha)(1-1/\psi)-1}}{((\theta+\alpha)(1-1/\psi)-\beta)^{(\theta+\alpha)(1-1/\psi)-\beta}} \right]^{-\psi} - 1 \right\}^{1/\psi} \tag{7}$$

Proof of Proposition 5.2. To apply Proposition 4.2, we need to ensure the existence of a root for $\pi_2(s,s) = 0$. To do this, note that $\pi_2(0,0) = -1$ and $\lim_{s\uparrow 1} \pi_2(s,s) = -\infty$. It suffices to show that there exists some $s \in (0,1)$ such that $\pi_2(s,s) > 0$. The first order derivative of $s \mapsto \pi_2(s,s)$ is $\frac{d}{ds}\pi_2(s,s) = \alpha\mu^{-\frac{1}{\psi}}((\theta+\alpha)(1-1/\psi)-1)s^{(\theta+\alpha)(1-1/\psi)-2} + (\beta-1)(1-s)^{\beta-2}$. By parametric conditions $(\theta+\alpha)(1-1/\psi) - 1 > 0$, $(\theta+\alpha)(1-1/\psi) < 2$ and $\beta < 1$, $\frac{d}{ds}\pi_2(s,s)$ is decreasing and has an interior root s_0. The interior root s_0 satisfies $\alpha\mu^{-\frac{1}{\psi}}((\theta+\alpha)(1-1/\psi)-1)s_0^{(\theta+\alpha)(1-1/\psi)-2} = (1-\beta)(1-s_0)^{\beta-2}$. Then plug it into the second term of $\pi_2(s_0,s_0)$ as follows,

$$\max_{s\in S} \pi_2(s,s) = \pi_2(s_0,s_0) = \alpha\mu^{-\frac{1}{\psi}} s_0^{(\theta+\alpha)(1-1/\psi)-2} \left[s_0 - \frac{1-s_0}{1-\beta}((\theta+\alpha)(1-1/\psi)-1) \right].$$

We want to show there exists some $s \in (0,1)$ such that $\pi_2(s,s) > 0$. It suffices to show $\max_{s\in S} \pi_2(s,s) > 0$, which is implied by $s_0 > \frac{(\theta+\alpha)(1-1/\psi)-1}{(\theta+\alpha)(1-1/\psi)-\beta}$. Thereby, we only need the strictly decreasing function $\frac{d}{ds}\pi_2(s,s)$ to attain a positive value at $\frac{(\theta+\alpha)(1-1/\psi)-1}{(\theta+\alpha)(1-1/\psi)-\beta}$, i.e.

$$\alpha\mu^{-\frac{1}{\psi}}((\theta+\alpha)(1-\frac{1}{\psi})-1)\left(\frac{(\theta+\alpha)(1-\frac{1}{\psi})-1}{(\theta+\alpha)(1-\frac{1}{\psi})-\beta}\right)^{(\theta+\alpha)(1-\frac{1}{\psi})-2}+(\beta-1)(1-\frac{(\theta+\alpha)(1-\frac{1}{\psi})-1}{(\theta+\alpha)(1-\frac{1}{\psi})-\beta})^{\beta-2}>0,$$

$$\iff \mu^{\frac{1}{\psi}}<\frac{\alpha}{(1-\beta)^{\beta-1}}\frac{((\theta+\alpha)(1-1/\psi)-1)^{(\theta+\alpha)(1-1/\psi)-1}}{((\theta+\alpha)(1-1/\psi)-\beta)^{(\theta+\alpha)(1-1/\psi)-\beta}},$$

By Proposition D.1, $\mu=\frac{1}{(b(1-1/\psi))^{\psi}+1}$, the inequality above entails

$$b>\frac{\psi}{\psi-1}\left\{\left[\frac{\alpha}{(1-\beta)^{\beta-1}}\frac{((\theta+\alpha)(1-1/\psi)-1)^{(\theta+\alpha)(1-1/\psi)-1}}{((\theta+\alpha)(1-1/\psi)-\beta)^{(\theta+\alpha)(1-1/\psi)-\beta}}\right]^{-\psi}-1\right\}^{1/\psi}. \tag{8}$$

Henceforth, $b>\bar{b}$ ensures the existence of $s\in\mathcal{S}$ such that $\pi_2(s,s)=0$.

For $\delta>0$, $\mathcal{L}(0,\delta)<0$ and $\mathcal{L}(1-,\delta)<0$. Then, by $\mathcal{L}(s,\delta)\geq\mathcal{L}(s,0)$, $\mathcal{L}(s,\delta)$ has two roots s_* and s^*. By Proposition 4.2, s^* is the unique interior locally stable steady state.

Here we show $\mathcal{B}(s^*,\delta)\supseteq\mathcal{B}(s_0,0)$. Denote the myopic basin of attraction $\mathcal{B}(s_0,0)$ by $[a,1]$. If $\Gamma(\cdot,\delta)$ is always above 45° to the left of s^* and always below 45° to the right, then s^* is globally stable. Since s_0 is stable, then by monotonicity, $\Gamma(\cdot,\delta)$ is still above 45° in $[a,s_0)$. Moreover, $\Gamma(\cdot,\delta)$ cannot be below 45° in $[s_0,s^*]$. If it were, by monotonicity and UHC, there would be a fixed point of $\Gamma(\cdot,\delta)$ in $[s_0,s^*)$. But that cannot be true because there are no more stable fixed points in the interior other than s^* by Proposition 3.2. For $(s^*,1]$, $\Gamma(\cdot,\delta)$ cannot jump above 45°. If it were, then there would be another stable steady state in $(s^*,1)$ or at 1. They can be both rejected by Proposition 3.2 and Lemma C.1.2 respectively. Thereby, we have shown $[a,1]\subseteq\mathcal{B}(s^*,\delta)$. □