Online Supplemental Material

Throughout this Supplemental Material, it is convenient to make the dependence on the discount factor, δ, explicit on $X[\bar{\Gamma}]$ and $X[\mathcal{L}]$ where X can be either $\mathcal{R}$ or $\mathcal{B}$. For this, we will use the notation $X[\bar{\Gamma}(\cdot,\delta)]$ and $X[\mathcal{L}(\cdot,\delta)]$.

SM.1 Unbounded State Space

It is possible to results in the text to the unbounded state space by imposing following technical assumption that is used to establish the existence of the value function.

Assumption SM.1.1. *There exists a continuous function $s \mapsto \varphi(s,\delta)$ such that $\varphi \geq c > 0$ and*

$$\sup_{s \geq 0} \sup_{s' \in \Upsilon(s)} \delta \frac{\varphi(s',\delta)}{\varphi(s,\delta)} < 1 \textit{ and } \sup_{s \geq 0} \sup_{s' \in \Upsilon(s)} \frac{\pi(s,s')}{\varphi(s,\delta)} < \infty.$$

If $\mathbb{S}$ is bounded, this assumption is vacuous (φ can be taken to be a constant). For unbounded state space, however, φ controls the growth of the per-period payoff and defines the relevant function space to which the value function belongs to: All $s \mapsto f(s)$ continuous such that $||f||_\delta := \sup_{s \in \mathbb{S}} |f(s)/\varphi(s,\delta)| < \infty$.[34]

For example, in the generalized NCG model, if one defines $\mathbb{S} = \mathbb{R}_+$, then Assumption SM.1.1 is satisfied with $s \mapsto \varphi(s,\delta) = u(f(s)) + A$, where A is such that $1 + \sup_{Y \in [0,1]} \frac{u'(Y)}{u(Y)+A}(f(Y) - Y)$ is strictly less than $1/\delta$.

Under Assumption SM.1.1 the results in the text go through essentially without change. The only change lies in Proposition 3.2 which continues to hold, but with the only caveat that $\mp\infty$ could be also steady states (see Proposition B.1 in Appendix B). Theorem 1, however, remains unchanged and so its refinements.

The proofs in the appendix will be done under this, more general, assumption.

[34]The concept underpinning Assumption SM.1.1 is by no means new, e.g. see Alvarez and Stokey (1998) for analogous conditions.

SM.2 Properties of the Value Function

Throughout this section we use

$$V \mapsto \mathbb{B}[V](s,\delta) := \max_{s' \in \Upsilon(s)} \pi(s,s') + \delta V(s',\delta) \;\forall (s,\delta) \in \mathbb{S} \times [0,1), \tag{9}$$

which is the Bellman operator defining the value function $(s,\delta) \mapsto V(s,\delta)$ as its fixed point.

Let $\mathbb{V} := \{f \colon \mathbb{S} \times [0,1) \to \mathbb{R} \colon f \textit{ is continuous and } f(.,\delta) \in \mathbb{V}_\delta \;\forall \delta \in [0,1)\}$, where $\mathbb{V}_\delta$ is the space of functions over $\mathbb{S}$ such that $||f||_\delta := \sup_{s\in\mathbb{S}} |f(s)/\varphi(s,\delta)| < \infty$. It is straightforward to show that $(\mathbb{V}_\delta, ||.||_\delta)$ is a Banach space.

Throughout, for any correspondence $s \mapsto F(s)$ let F^o denote the interior of it, i.e., for each s, $F^o(s)$ is the interior of the set $F(s)$.

SM.2.1 Existence and Uniqueness of the Value Function

Lemma SM.2.1. *$(s,\delta) \mapsto V(s,\delta)$ is the unique solution to $V = \mathbb{B}[V]$ in the space $\mathbb{V}$.*

Proof of Lemma SM.2.1. For each $\delta \in [0,1)$, we show that the Bellman operator, $\mathbb{B}_\delta$ — acting on functions of s, not δ — maps $\mathbb{V}_\delta$ into itself and it is a contraction. By Assumption SM.1.1, $(s,s') \mapsto \pi(s,s')/\varphi(s,\delta)$ is bounded over $\mathit{Graph}\Upsilon$. Hence, $\mathbb{B}_\delta$ maps bounded function into itself. By continuity of π and the fact that $s \mapsto \Upsilon(s)$ is continuous and compact-valued (see Assumption 2), the ToM implies that $\mathbb{B}_\delta$ maps $\mathbb{V}_\delta$ into itself. The contraction property follows from standard Blackwell sufficient conditions.

Therefore, for each $\delta \in [0,1)$, there exists a unique function on $\mathbb{V}_\delta$, denoted as $V(\cdot,\delta)$, that is a fixed point of $\mathbb{B}_\delta$. Now define $(s,\delta) \mapsto V(s,\delta) := V(s,\delta)$. Clearly, $V \in \mathbb{V}$ and is the unique function such that $V(.,\delta) = \mathbb{B}_\delta[V(.,\delta)]$ for all $\delta \in [0,1)$, but this readily implies that $V = \mathbb{B}[V]$. $\square$

SM.2.2 Monotonicity of the Value Function

In this section we present three approaches to establish that $s \mapsto V(s,\delta)$ is increasing, each relying on different assumptions and different proof techniques. One, relies on $s \mapsto$

$\Upsilon(s)$ being increasing in the sense of inclusion and is standard (cf. Stokey et al. (1989)). However, this condition on Υ might be too strong in some settings. So, we also present two alternative approaches, which dispenses with the aforementioned assumption. One relies on weaker assumptions — namely, on being able to 'replicate' payoffs generated for different values of the state variables —, the other one relies on a generalized version of the mean value theorem for a.e. smooth functions. To our knowledge, these last two approaches are novel and might be of independent interest.

Here are the results under the first approach; it is a well-known result and it is here merely for completeness.

Lemma SM.2.2. *Suppose $s \mapsto \Upsilon(s)$ is non-decreasing in the inclusion sense. Then $s \mapsto V(s,\delta)$ is increasing.*

Proof of Lemma SM.2.2. We first show that $\mathbb{B}_\delta$ maps non-decreasing functions into themselves. To do this take any $f \in \mathbb{V}_\delta$ that is non-decreasing. Then

$$\mathbb{B}_\delta[f](s_0) \geq \pi(s_0, s') + \delta f(s') > \pi(s_1, s') + \delta f(s')$$

for any $s_0 > s_1$ and any $s' \in \Upsilon(s_0)$; the second inequality follows from the fact that $s \mapsto \pi(s, s')$ is increasing (Assumption 1(i)).

By assumption, $\Upsilon(s_0) \supseteq \Upsilon(s_1)$, so s' can be taken to be in $\Gamma_\delta(s_1)$ and thus obtain $\mathbb{B}_\delta[f](s_0) > \mathbb{B}_\delta[f](s_1)$. Thus, $s \mapsto \mathbb{B}_\delta[f](s)$ is increasing, and so $\mathbb{B}_\delta$ maps non-decreasing functions into increasing ones (and thus non-decreasing). Since the class of non-decreasing is closed, it follows that $V(\cdot,\delta)$ is non-decreasing. Indeed, since $V(\cdot,\delta)$ is unique, it actually is increasing. $\square$

Here is the second approach

Lemma SM.2.3. *Suppose for any $s_1 > s_0$ there exists a pair s'_1, s'_0 with $s'_1 \in \Upsilon(s_1)$ and $s'_0 \in \Gamma(s_0,\delta)$ such that (i) $s'_1 \geq s'_0$ and (ii) $\pi(s_1, s'_1) \geq \pi(s_0, s'_0)$ with at least one of the inequalities strict. Then $s \mapsto V(s,\delta)$ is increasing.*

Proof of Lemma SM.2.3. We first show that $\mathbb{B}_\delta$ maps non-decreasing functions into themselves. To do this take any $f \in \mathbb{V}_\delta$ that is non-decreasing. Then for any $s_1 > s_0$,

$$\mathbb{B}_\delta[f](s_1) \geq \pi(s_1, s_1') + \delta f(s_1') \geq \pi(s_0, s_0') + \delta f(s_1') \geq \pi(s_0, s_0') + \delta f(s_0') = \mathbb{B}_\delta[f](s_0),$$

where the first inequality holds because $s_1' \in \Upsilon(s_1)$; the second inequality follows from condition (ii); the third inequality follows from condition (i) and the fact that f is non-decreasing; the last equality follows because $s_0' \in \Gamma(s_0, \delta)$.

Thus, $s \mapsto \mathbb{B}_\delta[f](s)$ is increasing, and so $\mathbb{B}_\delta$ maps non-decreasing functions into increasing ones (and thus non-decreasing). Since the class of non-decreasing is closed, it follows that $s \mapsto V(s, \delta)$ is non-decreasing. Indeed, since one of the inequalities is strict and $s \mapsto V(s, \delta)$ is unique, it actually is increasing. □

Remark SM.2.1. (1) The condition in the lemma is implied by increasing in inclusion sense. To see this, suppose $\Upsilon(s_1) \supseteq \Upsilon(s_0)$ then take $s_1' = s_0'$ with any $s_0' \in \Gamma(s_0, \delta)$. Then clearly (i) is met and (ii) follows as $\pi_1 > 0$. By the inclusion, $s_1' \in \Upsilon(s_1)$.

(2) The condition is met in the generalized NCG model. To see this, let $s_1' = -\bar{f}(s_0) + \bar{f}(s_1) + s_0'$ where $\bar{f}(s) = f(s) + (1-d)s$. (i) holds as $-\bar{f}(s_0) + \bar{f}(s_1) > 0$; (ii) holds as $u(\bar{f}(s_1) - s_1') = u(\bar{f}(s_0) - s_0')$. We need to verify $s_1' \in \Upsilon(s_1) = [(1-d)s_1, (1-d)s_1 + f(s_1)]$. To check this, note that

$$s_0' \in \Upsilon(s_0) \implies s_0' \leq \bar{f}(s_0) \iff \bar{f}(s_1) - \left[\bar{f}(s_0) - s_0'\right] \leq \bar{f}(s_1) \implies s_1' \leq \bar{f}(s_1)$$

and

$$\begin{aligned} s_0' \in \Gamma(s_0, \delta) \implies s_0' \geq (1-d)\, s_0 \implies f(s_1) - f(s_0) + \left[s_0' - (1-d)\, s_0\right] \geq 0 \\ \iff \bar{f}(s_1) - \left[\bar{f}(s_0) - s_0'\right] \geq (1-d)\, s_1 \iff s_1' \geq (1-d)\, s_1 \end{aligned}$$

△

Here are the results under the third approach. Given the nature of the proof, it requires conditions that ensure optimal choices are interior.

Lemma SM.2.4. *For any $s_0 < s_1$ in $\mathbb{S}$ such that $\Gamma(s, \delta) \subseteq \Upsilon^o(s)$ for all $s \in (s_0, s_1)$, there exists* *a* $c \in (s_0, s_1)$ *such that*

$$\max_{Y \in \Gamma(c,\delta)} \partial_1^+ \pi(c, Y) = \max\{\partial^+ V(c,\delta), \partial^- V(c,\delta)\}$$
$$\geq \frac{V(s_1,\delta) - V(s_0,\delta)}{s_1 - s_0} \geq$$
$$\min\{\partial^+ V(c,\delta), \partial^- V(c,\delta)\} = \min_{Y \in \Gamma(c,\delta)} \partial_1^- \pi(c, Y)$$

Proof. Let $\frac{V(s_1,\delta)-V(s_0,\delta)}{s_1-s_0} =: D$. By construction, $G(s) := V(s,\delta) - Ds$ is such that $G(s_1) = G(s_0)$. Since G is continuous, the function G achieves a minimum or a maximum in (s_0, s_1); denote the point as c.

By assumption, $\Gamma(c,\delta) \subseteq \mathrm{Y}^o(c)$, so by Proposition 3.1, the left and right derivatives of $s \mapsto V(s,\delta)$ (and thus of G) exist. If c is a minimizer then $\partial^- G(c) \leq 0 \leq \partial^+ G(c)$, and if c is a maximizer then $\partial^- G(c) \geq 0 \geq \partial^+ G(c)$. These inequalities imply that $\partial^- V(c,\delta) \leq D \leq \partial^+ V(c,\delta)$ and $\partial^- V(c,\delta) \geq D \geq \partial^+ V(c,\delta)$ respectively. Hence,

$$\min\{\partial^+ V(c,\delta), \partial^- V(c,\delta)\} \leq D \leq \max\{\partial^+ V(c,\delta), \partial^- V(c,\delta)\}.$$

This implies

$$\min\{\partial^+ V(c,\delta), \partial^- V(c,\delta)\} \leq \frac{V(s_1,\delta) - V(s_0,\delta)}{s_1 - s_0} \leq \max\{\partial^+ V(c,\delta), \partial^- V(c,\delta)\}.$$

By Proposition 3.1 and the fact that $\min_{Y \in \Gamma(c,\delta)} \partial_1^- \pi(c, Y) \leq \max_{Y \in \Gamma(c,\delta)} \partial_1^+ \pi(c, Y)$ (Assumption 1(i)), the outer equality in the lemma follows. □

Corollary SM.2.1. *For any* $s_0 < s_1$ *in* $\mathbb{S}$ *such that* $\Gamma(s,\delta) \subseteq \mathrm{Y}^o(s)$ *for all* $s \in (s_0, s_1)$*,* $V(s_1,\delta) > V(s_0,\delta)$*.*

Proof. By Lemma SM.2.4, $V(s_1,\delta) - V(s_0,\delta) \geq \min_{Y \in \Gamma(c,\delta)} \partial_1^- \pi(c, Y)(s_1 - s_0)$. By Assumption 1(i) $\partial_1^- \pi(c, Y) > 0$ so the desired result follows. □

SM.2.3 Differentiability of the Value Function

Lemma SM.2.5. *For any* $s \in Range(\Gamma_\delta \cap \mathrm{Y}^o)$ *the following are true:*

1. *$s \mapsto V(s,\delta)$ is smooth. That is, $\partial^+ V(s,\delta) = \partial^- V(s,\delta) = \pi_1(s,s')$ for any $s' \in \Gamma(s,\delta)$.*

2. *$\Gamma(s,\delta)$ is a singleton and $\partial_1^+ \pi(s,\Gamma(s,\delta)) = \partial_1^- \pi(s,\Gamma(s,\delta)) =: \pi_1(s,\Gamma(s,\delta))$.*

Proof of Lemma SM.2.5. For any $s \in Range\Gamma(\cdot,\delta)$, it follows that there exists a s_{-1} such that $s \in \Gamma(s_{-1},\delta)$. Hence, $\pi(s_{-1},s) + \delta V(s,\delta) \geq \pi(s_{-1},s') + \delta V(s',\delta)$ for any $s' \in \Upsilon(s_{-1})$. Since $s \in \Upsilon^o(s_{-1})$, the previous inequality holds for $s' = s + \Delta$ and $s' = s - \Delta$ for sufficiently small $\Delta > 0$. By Proposition 3.1, $\partial_+ V(s,\delta)$ exists and thus

$$0 \geq \pi_2(s_{-1},s) + \delta\partial^+ V(s,\delta) \textit{ and } 0 \leq \pi_2(s_{-1},s) + \delta\partial^- V(s,\delta).$$

The two inequalities imply that $\partial^- V(s,\delta) \geq \partial^+ V(s,\delta)$.

On the other hand, by Proposition 3.1 and Assumption 1(i), $\partial^- V(s,\delta) = \min_{Y\in\Gamma(s,\delta)} \partial_1^- \pi(s,Y) \leq \partial_1^- \pi(s,Y) \leq \partial_1^+ \pi(s,Y) \leq \max_{y\in\Gamma(s,\delta)} \partial_1^+ \pi(s,Y) = \partial^+ V(s,\delta)$.

Therefore, $\partial^- V(s,\delta) = \partial^+ V(s,\delta) = \partial_1^+ \pi(s,Y) = \partial_1^- \pi(s,Y)$ for any $Y \in \Gamma(s,\delta)$.

We conclude the proof by showing that $\Gamma(s,\delta)$ is a singleton. This readily follows by the previous result which imply that $\min_{Y\in\Gamma(s,\delta)} \pi_1(s,Y) = \max_{Y\in\Gamma(s,\delta)} \pi_1(s,Y)$, which by Assumption 1(iii) can only hold if $\Gamma(s,\delta)$ is a singleton. □

Lemma SM.2.6. *$\delta \mapsto V(s,\delta)$ is continuously differentiable and for any $s \in \mathbb{R}_+$,*

$$\frac{dV(s,\delta)}{d\delta} = \sum_{t=0}^{\infty} \delta^t V(s^0_{t+1},\delta) = V(s_1,\delta) + \delta\frac{dV(s_1,\delta)}{d\delta}.$$

for some $(s_t)_t$ such that $s_0 = s$ and $s_{t+1} \in \Gamma(s_t,\delta)$.

Proof of Lemma SM.2.6 . For any $s \geq 0$ and $\delta \in [0,1)$ and $\Delta \in \mathbb{R}$,

$$V(s,\delta+\Delta) - V(s,\delta) \geq \delta(V(s_1,\delta+\Delta) - V(s_1,\delta)) + \Delta V(s_1,\delta+\Delta)$$

for any $s_1 \in \Gamma(s,\delta)$. Applying this again to s_1, it follows that

$$V(s,\delta+\Delta) - V(s,\delta) \geq \delta^2(V(s_2,\delta+\Delta) - V(s_2,\delta)) + \Delta(V(s_1,\delta+\Delta) - \pi(s,s_1)) + \Delta\delta V(s_2,\delta+\Delta)$$

for any $s_2 \in \Gamma(s_1,\delta)$.

By iterating in this fashion, one obtains for any $T > 1$,

$$V(s, \delta + \Delta) - V(s, \delta) \geq \delta^{T+1}(V(s_{T+1}, \delta + \Delta) - V(s_{T+1}, \delta)) + \Delta \sum_{t=0}^{T} \delta^t V(s_{t+1}, \delta + \Delta)$$

where $s_0 := s$ and any $(s_t)_t$ such that $s_{t+1} \in \Gamma(s_t, \delta)$.

We now show that $\lim_{T\to\infty} \delta^{T+1}(V(s_{T+1}, \delta + \Delta) - V(s_{T+1}, \delta)) = 0$. If $\mathbb{S}$ is bounded this is trivial because V is bounded and $\delta < 1$. In the more general case, under Assumption SM.1.1, it follows that there exists a constant K and a $0 \leq a < 1$ such that for any s_t, $V(s_t, \delta) = \frac{V(s_t,\delta)}{\varphi(s_t,\delta)} \varphi(s_t, \delta) \leq K\varphi(s_t, \delta)$ and $\varphi(s_t, \delta) \leq (a/\delta)\varphi(s_{t-1}, \delta)$. Thus, iterating in this fashion it follows that $V(s_t, \delta) \leq K(a/\delta)^t \varphi(s_0, \delta) = O((a/\delta)^t)$. Therefore,

$$\delta^{T+1}(V(s_{T+1}, \delta + \Delta) - V(s_{T+1}, \delta)) = O(a^{T+1}),$$

and since $0 \leq a < 1$ the desired result follows.

Therefore,

$$V(s, \delta + \Delta) - V(s, \delta) \geq \Delta \sum_{t=0}^{\infty} \delta^t V(s_{t+1}, \delta + \Delta).$$

Taking $\Delta > 0$ and taking limits it follows that

$$\partial_+ V(s, \delta) := \limsup_{\Delta \downarrow 0} \frac{V(s, \delta + \Delta) - V(s, \delta)}{\Delta} \geq \limsup_{\Delta \to 0} \sum_{t=0}^{\infty} \delta^t V(s_{t+1}, \delta + \Delta).$$

Also, taking $\Delta < 0$ and taking limits it follows that,

$$\partial_- V(s, \delta) := \liminf_{\Delta \uparrow 0} \frac{V(s, \delta + \Delta) - V(s, \delta)}{\Delta} \leq \liminf_{\Delta \to 0} \sum_{t=0}^{\infty} \delta^t V(s_{t+1}, \delta + \Delta).$$

By Lemma SM.2.1, $\delta \mapsto V(s, \delta)$ is continuous. Hence since $\sum_{t=0}^{\infty} \delta^t V(s_{t+1}, \delta + \Delta) = O(\sum_{t=0}^{\infty} a^t) < \infty$, by DCT, it follows that $\lim_{\Delta\to 0} \sum_{t=0}^{\infty} \delta^t V(s_{t+1}, \delta + \Delta) = \sum_{t=0}^{\infty} \delta^t V(s_{t+1}, \delta)$ and so

$$\partial_+ V(s, \delta) \geq \sum_{t=0}^{\infty} \delta^t V(s_{t+1}, \delta) \geq \partial_- V(s, \delta), \tag{10}$$

for any $(s_t)_t$ such that $s_0 = s$ and $s_{t+1} \in \Gamma(s_t, \delta)$.

Similarly,

$$V(s, \delta + \Delta) - V(s, \delta) \leq \delta(V(s_1, \delta + \Delta) - V(s_1, \delta)) + \Delta V(s_1, \delta + \Delta)$$

for any $s_1 \in \Gamma(s, \delta + \Delta)$. By iterating in the same way as before, it follows that

$$V(s, \delta + \Delta) - V(s, \delta) \leq \Delta \sum_{t=0}^{\infty} \delta^t V(s_{t+1}, \delta + \Delta)$$

for any $(s_t)_t$ such that $s_0 = s$ and $s_{t+1} \in \Gamma(s_t, \delta + \Delta)$.

For each Δ take $s_1 =: s_1^\Delta$ such that $s_1^\Delta \in \Gamma(s, \delta + \Delta)$ and $\lim_{\Delta \to 0} s_1^\Delta = s_1^0$; this last property holds because $s \in [0, 1]$ and any sequence has a convergent subsequence. By Lemma SM.3.1, $\delta \mapsto \Gamma(s, \delta)$ is UHC, so $s_1^0 \in \Gamma(s, \delta)$. Now take $s_2 := s_2^\Delta$ such that $s_2^\Delta \in \Gamma(s_1^\Delta, \delta + \Delta)$ and $\lim_{\Delta \to 0} s_2^\Delta = s_2^0$. By Lemma SM.3.1, $(s, \delta) \mapsto \Gamma(s, \delta)$ is UHC; this fact and the fact that $\lim_{\Delta \to 0} s_1^\Delta = s_1^0$, imply that $s_2^0 \in \Gamma(s_1^0, \delta)$. By iterating in this fashion we obtain a sequence $(s_t^\Delta)_t$ such that pointwise in t, $\lim_{\Delta \to 0} s_t^\Delta = s_t^0$ and $s_{t+1}^0 \in \Gamma(s_t^0, \delta)$.

By continuity of π, $\lim_{\Delta \to 0} \pi(s_t^\Delta, s_{t+1}^\Delta) = \pi(s_t^0, s_{t+1}^0)$ (pointwise on t). By Lemma SM.2.1, $(s, \delta) \mapsto V(s, \delta)$ is continuous, so $\lim_{\Delta \to 0} V(s_{t+1}^\Delta, \delta + \Delta) = V(s_{t+1}^0, \delta)$ (pointwise on t). Therefore,

$$\lim_{\Delta \to 0} V(s_{t+1}^\Delta, \delta + \Delta) = V(s_{t+1}^0, \delta).$$

Since $\sum_{t=0}^{\infty} \delta^t < \infty$ and $V(\cdot, \delta)$ is also uniformly bounded, it follows by the DCT that

$$\partial_+ V(s, \delta) := \limsup_{\Delta \downarrow 0} \frac{V(s, \delta + \Delta) - V(s, \delta)}{\Delta} \leq \limsup_{\Delta \downarrow 0} \sum_{t=0}^{\infty} \delta^t V(s_{t+1}^\Delta, \delta + \Delta) \leq \sum_{t=0}^{\infty} \delta^t V(s_{t+1}^0, \delta).$$

Analogous reasoning to that above yields

$$\partial_- V(s, \delta) \geq \limsup_{\Delta \downarrow 0} \sum_{t=0}^{\infty} \delta^t V(s_{t+1}^\Delta, \delta + \Delta) \geq \sum_{t=0}^{\infty} \delta^t V(s_{t+1}^0, \delta).$$

Combining these displays with the expression in 10, it follows that

$$\partial_+ V(s,\delta) = \sum_{t=0}^{\infty} \delta^t V(s_{t+1},\delta) = \partial_- V(s,\delta), \tag{11}$$

for some $(s_t)_t$ such that $s_0 = s$ and $s_{t+1} \in \Gamma(s_t,\delta)$. $\square$

SM.3 Properties of the optimal correspondence

Recall

$$(s,\delta) \mapsto \Gamma(s,\delta) := \arg\max_{s' \in \Upsilon(s)} \pi(s,s') + \delta V(s',\delta).$$

SM.3.1 Topological Properties of the Optimal Correspondence

Lemma SM.3.1. *$(s,\delta) \mapsto \Gamma(s,\delta)$ is non-empty, compact-valued, and UHC.*

Proof of Lemma SM.3.1. By Lemma SM.2.1 and Assumptions 1 and 2, $(s,\delta,s') \mapsto \pi(s,s') + \delta V(s',\delta)$ is continuous and $s \mapsto \Upsilon(s)$ is compact-valued and continuous. Thus, by the ToM, $(s,\delta) \mapsto \Gamma(s,\delta)$ is non-empty, compact-valued, and UHC. $\square$

The next lemma essentially shows that Γ_δ is a function at fixed points.

Lemma SM.3.2. *If $s \in \Gamma(s,\delta) \cap \Upsilon^o(s)$, then $s = \Gamma(s,\delta)$.*

Proof of Lemma SM.3.2. Observe that s trivially belongs to *Range*$\Gamma(\cdot,\delta) \cap \Upsilon^o$. Then, by Lemma SM.2.5 $\Gamma(s,\delta)$ is a singleton thereby implying the desired result. $\square$

SM.3.2 Monotonicity of the Optimal Correspondence

Recall that a correspondence is function-like if its graph has an empty interior.

Lemma SM.3.3. *$s \mapsto \Gamma(s,\delta)$ is non-decreasing — in the sense that* $\max \Gamma(a,\delta) \leq \min \Gamma(b,\delta)$ *for any $a < b$ — and function-like.*

Proof of Lemma SM.3.3. Observe that $(s,s') \mapsto F(s,s') := \pi(s,s') + \delta V(s',\delta)$ satisfies increasing differences since for any $s_0 > s_1$, $D(s') := F(s_0,s') - F(s_1,s') = \pi(s_0,s') -$

$\pi(s_1, s')$. Taking the derivative with respect to s' it follows that $D'(s') = \pi_2(s_0, s') - \pi_2(s_1, s')$ which is positive by Assumption 1(iii). By Assumption 2, Y is non-decreasing in the strong set order sense. Hence, by the Milgrom-Shannon Theorem (Milgrom and Shannon (1994)), for any $a < b$ and any $y \in \Gamma(a, \delta)$ and $y' \in \Gamma(b, \delta)$ it follows that $\min\{y, y'\} \in \Gamma(a, \delta)$ and $\max\{y, y'\} \in \Gamma(b, \delta)$.

We now show that $\max \Gamma(a, \delta) \leq \min \Gamma(b, \delta)$. Suppose not, suppose there exists a $y \in \Gamma(a, \delta)$ and $y' \in \Gamma(b, \delta)$ such that $y' < y$. By the previous result, $y' \in \Gamma(a, \delta)$ and $y \in \Gamma(b, \delta)$. So, $y, y' \in \Gamma(a, \delta) \cap \Gamma(b, \delta)$ which implies that

$$\pi(b, y) + \delta V(y, \delta) = \pi(b, y') + \delta V(y', \delta) \text{ and } \pi(a, y) + \delta V(y, \delta) = \pi(a, y') + \delta V(y', \delta).$$

This implies that $\pi(b, y) - \pi(b, y') = \pi(a, y) - \pi(a, y')$. By the mean value theorem, this equality implies

$$\int_0^1 \pi_2(b, \tau y' + (1 - \tau) y) d\tau (y - y') = \int_0^1 \pi_2(a, \tau y' + (1 - \tau) y) d\tau (y - y').$$

However, by Assumption 1(iii), $s \mapsto \pi_2(s, s')$ is increasing, so $\pi_2(a, \tau y' + (1 - \tau) y) < \pi_2(b, \tau y' + (1 - \tau) y)$ for any τ and $a < b$. Therefore, the previous display cannot hold thereby yielding a contradiction.

We now show that $\Gamma(\cdot, \delta)$ is function-like. Otherwise, there exists a $(s, y) \in Graph \Gamma(\cdot, \delta)$ that has an open neighborhood also in the graph. Thus one can find a pair $(s + a, y - a)$ with $a > 0$ such that $\Gamma(s + a, \delta) \ni y - a$. But we found $y - a \in \Gamma(s + a, \delta)$ and $y \in \Gamma(s, \delta)$ such that $y - a = \min\{y, y - a\} \in \Gamma(s + a, \delta)$ which contradicts the previous results. $\square$

Recall that $\delta \mapsto \Gamma(s, \delta)$ is non-decreasing (in the strong set sense) if for any $\delta' > \delta$, any $s \in \mathbb{R}_+$ and any $y \in \Gamma(s, \delta)$ and $y' \in \Gamma(s, \delta')$ it follows that $\min\{y, y'\} \in \Gamma(s, \delta)$ and $\max\{y, y'\} \in \Gamma(s, \delta')$.

Lemma SM.3.4. *Take a $s \in \mathbb{S}$ such that $\Gamma(s, \delta) \subseteq Y^o(s)$ and suppose*[35]

$$V(\cdot, \delta) \text{ is increasing over } \cup_{l=0}^{\infty} \Gamma^l(Y^o(s), \delta). \tag{12}$$

[35] The set $\Gamma^1_\delta(Y^o(s)) := \{\Gamma(s', \delta) \colon s' \in Y^o(s)\}$, and $\Gamma^l_\delta(s) := \Gamma(\Gamma^{l-1}_\delta(s), \delta)$ for any $l > 1$. For $l = 0$, $\Gamma^0_\delta(Y^o(s)) := Y^o(s)$.

Then, for any $(y, y') \in \Gamma(s, \delta) \times \Gamma(s, \delta')$ *it follows that* $\max\{y, y'\} \in \Gamma(s, \delta')$ *and* $\min\{y, y'\} \in \Gamma(s, \delta)$ *for any* $\delta' > \delta$.

Moreover,[36]

$$\min \Gamma(s, \delta') \geq \max \Gamma(s, \delta). \tag{13}$$

Proof of Lemma SM.3.4. Let $(s', \delta) \mapsto W(s', \delta) := \delta V(s', \delta)$. Since s is such that $\Gamma(s, \delta) \subseteq \Upsilon^o(s)$, it follows that $\Gamma(s, \delta) = \arg\max_{s' \in \Upsilon(s)} \pi(s, s') + W(s', \delta) = \arg\max_{s' \in \Upsilon^o(s)} \pi(s, s') + W(s', \delta)$. By Milgrom-Shannon theorem, if W is (a) quasi-super modular and (b) has the single-crossing property then $\delta \mapsto \Gamma(s, \delta)$ is non-decreasing. Thus, we need to establish properties (a) and (b). For this it suffices to show that $s' \mapsto \frac{dW(s',\delta)}{d\delta}$ is increasing. Observe that, by Lemma SM.2.6,

$$s' \mapsto \frac{dW(s', \delta)}{d\delta} = V(s', \delta) + \delta \frac{dV(s', \delta)}{d\delta} = V(s', \delta) + \delta \left(V(s'', \delta) + \delta \frac{dV(s'', \delta)}{d\delta} \right),$$

for some $s'' \in \Gamma(s', \delta)$.

Now consider $s'_1 > s'_0$ in $\Upsilon^o(s)$. By the condition in the lemma, $V(s'_0, \delta) < V(s'_1, \delta)$. By Lemma SM.3.3, $s \mapsto \Gamma(s, \delta)$ is non-decreasing so $s''_0 \leq s''_1$ for any $s''_l \in \Gamma(s'_l, \delta)$ for $l = 0, 1$. Thus $V(s''_0, \delta) \leq V(s''_1, \delta)$ by our assumption as s''_0, s''_1 belong to $\Gamma^2(s, \delta)$. By Lemma SM.2.6, iterating in this fashion establishes that $\frac{dV(s'_0,\delta)}{d\delta} \leq \frac{dV(s'_1,\delta)}{d\delta}$. Thus $\frac{dW(s'_0,\delta)}{d\delta} < \frac{dW(s'_1,\delta)}{d\delta}$ as desired.

We now show that $\min \Gamma(s, \delta') \geq \max \Gamma(s, \delta)$ for any $\delta' > \delta$. We do this by contradiction, i.e., suppose there exists an $a \in \Gamma(s, \delta')$ and a $b \in \Gamma(s, \delta)$ such that $a < b$. By the first part of this lemma, $b \in \Gamma(s, \delta')$ and $a \in \Gamma(s, \delta)$; so, $a, b \in \Gamma(s, \delta') \cap \Gamma(s, \delta)$. This implies that

$$\pi(s, a) + \delta V(a, \delta) = \pi(s, b) + \delta V(b, \delta) \textit{ and } \pi(s, a) + \delta' V(a, \delta') = \pi(s, b) + \delta' V(b, \delta')$$

thus, by re-arranging terms it follows that $W(a, \delta') - W(a, \delta) = W(b, \delta') - W(b, \delta)$. By the

[36]For a set S, $\min S := \min\{s \colon s \in S\}$ and analogously with $\max S$.

mean value theorem and the fact that $\delta' \neq \delta$, this implies that

$$\int_0^1 \frac{dW(a, \delta + t(\delta' - \delta)))}{d\delta} dt(\delta' - \delta) = \int_0^1 \frac{dW(b, \delta + t(\delta' - \delta)))}{d\delta} dt(\delta' - \delta).$$

However, we established that $y \mapsto \frac{dW(y,.)}{d\delta}$ was increasing and since $a < b$ this implies that $\frac{dW(a,\delta+t(\delta'-\delta)))}{d\delta} < \frac{dW(b,\delta+t(\delta'-\delta)))}{d\delta}$ for all t, a contradiction. □

Remark SM.3.1. The condition in the lemma is a high level condition that can be verified in many different ways. For instance, if Y is non-decreasing in the inclusion sense then by Lemma SM.2.2 the condition immediately follows.

If Y is not non-decreasing in the inclusion sense, the condition in the lemma can still be verified following the results in Appendix SM.2.2. △

SM.3.3 Characterization of Basins of Attraction of Γ

Recall that for any function $F : \mathbb{S} \to \mathbb{R}$ and for any $e \in \mathbb{S}$ let

$$\mathcal{B}^{+}[F](e) := \{s \in \mathbb{S} \colon s < e \text{ and } \forall s' \in (s,e)\ F(s') > 0\} \cup \{e\}$$
$$\text{and } \mathcal{B}^{-}[F](e) := \{s \in \mathbb{S} \colon s > e \text{ and } \forall s' \in (e,s)\ F(s') < 0\} \cup \{e\}.$$

That is, $\mathcal{B}^{+}[F](e)$ is the set of all points $s \in \mathbb{S}$ such that for any point between s and e, the function $s \mapsto F(s)$ is positive — this set includes e as a convention that simplifies the exposition.

We refer to $\mathcal{B}[F](e) = \mathcal{B}^{-}[F](e) \cup \mathcal{B}^{+}[F](e)$ as the *Basin of attraction (BoA) for F*; this terminology is justified by the following result.

Proposition SM.3.1. *For any $e \in \mathcal{R}^{o}[\bar{\Gamma}(\cdot,\delta)]$, $\mathcal{B}(e,\delta) = \mathcal{B}[\bar{\Gamma}(\cdot,\delta)](e)$.*

Proof of Proposition SM.3.1. Let $l(e) := \inf\{s \colon s \in \mathcal{B}^{+}[\bar{\Gamma}(\cdot,\delta)](e)\}$. Clearly, $\mathcal{B}^{+}[\bar{\Gamma}(\cdot,\delta)](e) = (l(e), e]$. Similarly, let $u(e) := \sup\{s \colon s \in \mathcal{B}^{-}[\bar{\Gamma}(\cdot,\delta)](e)\}$. Clearly, $\mathcal{B}^{-}[\bar{\Gamma}(\cdot,\delta)](e) = [e, u(e))$.

We first show that $\mathcal{B}(e,\delta) \supseteq \mathcal{B}^{+}[\bar{\Gamma}(\cdot,\delta)](e) \cup \mathcal{B}^{-}[\bar{\Gamma}(\cdot,\delta)](e)$. To do this, take any $s \in \mathcal{B}^{+}[\bar{\Gamma}(\cdot,\delta)](e)$. By definition $s > l(e)$ and $\Gamma(s,\delta) > s$. Therefore, since $s \mapsto \Gamma(s,\delta)$ is non-decreasing (Lemma SM.3.3 in Appendix SM.3) any flow starting in s is such that

$\phi_\delta(t,s) \geq \phi_\delta(t-1,s)$ for any $t \in \mathbb{N}$. Therefore it must have a limit point which we denote as a. Since $s \leq e$, $\phi_\delta(t,s) \leq \phi_\delta(t,e) = e$, which implies that $a \leq e$. If $a < e$ it means that a is a fixed point of $\Gamma(\cdot,\delta)$ that is in (s,e), but this contradicts the definition of $\mathcal{B}^+[\bar{\Gamma}(\cdot,\delta)](e)$. Thus $a = e$ thereby showing that $\mathcal{B}(e,\delta) \supseteq \mathcal{B}^+[\bar{\Gamma}(\cdot,\delta)](e)$. Analogous arguments can show that $\mathcal{B}(e,\delta) \supseteq \mathcal{B}^-[\bar{\Gamma}(\cdot,\delta)](e)$ and thus the desired inclusion holds.

We now show that $\mathcal{B}(e,\delta) \subseteq \mathcal{B}^+[\bar{\Gamma}(\cdot,\delta)](e) \cup \mathcal{B}^-[\bar{\Gamma}(\cdot,\delta)](e)$. We do this by contradiction, i.e., suppose there exists a $s_0 \in \mathcal{B}(e,\delta)$ but $s_0 \notin \mathcal{B}^+[\bar{\Gamma}(\cdot,\delta)](e) \cup \mathcal{B}^-[\bar{\Gamma}(\cdot,\delta)](e)$. This means that either (a) $s_0 < e$ but there exists a $a \in \Gamma(s_0,\delta)$ such that $a \leq s_0$; or $s_0 > e$ but there exists a $a \in \Gamma(s_0,\delta)$ such that $a \geq s_0$.

Suppose (a) holds. Clearly, $s_0 = a$ is a direct contradiction to the fact that $s_0 \in \mathcal{B}(e,\delta)$, so lets take $a < s_0$. This means there exist at least one flow such that $\phi(1,s_0) \leq s_0$. By Lemma SM.3.3 in Appendix SM.3 this implies that $\phi(t,s_0) \leq s_0$ for all t. But since $e > s_0$, this implies that $(\phi(t,s_0))_t$ does not converge to e, contradicting the assumption that $s_0 \in \mathcal{B}(e,\delta)$. If (b) holds analogous arguments also show a contradiction to the assumption that $s_0 \in \mathcal{B}(e,\delta)$.

Hence, $\mathcal{B}(e,\delta) \subseteq \mathcal{B}^+[\bar{\Gamma}(\cdot,\delta)](e) \cup \mathcal{B}^-[\bar{\Gamma}(\cdot,\delta)](e)$, thereby establishing the desired result. □

SM.4 Sufficient Conditions for Locator functions to have well-separated roots

Lemma SM.4.1. *Suppose the locator function is three times continuously differentiable and*

- *For any s such that $\mathcal{L}_1(s,\delta) = 0$, then $|\mathcal{L}_{11}(s,\delta)| > 0$.*

Then for any root r, there exists an open neighborhood such that no other roots are in it.

If, in addition, there exists a compact set $S \subseteq \mathbb{S}$ such that $|\mathcal{L}(s,\delta)| > 0$ for all $s \notin S$, then the roots are finite.

Proof. We say s is a critical point of the locator function if $\mathcal{L}_1(s,\delta) = 0$. We now show that critical points are isolated, i.e., for any critical point s, there exists a $\epsilon_s > 0$ such that $\mathcal{L}_1(s,\delta) \neq 0$ for all $s \in (s - \epsilon_s, s + \epsilon_s)$. We prove this by contradiction, i.e., suppose that there exists a sequence $(c_n)_n$ of critical points that converge to a distinct critical point c. Then by the MVT (and the fact that the locator function is three times differentiable over a bounded domain)

$$\mathcal{L}_1(c_n, \delta) - \mathcal{L}_1(c, \delta) = \mathcal{L}_{11}(c, \delta)(c_n - c) + o(c_n - c),$$

but the LHS is zero as both are critical points, thereby implying that $0 = \mathcal{L}_{11}(c, \delta) + \frac{o(c_n - c)}{(c_n - c)}$. Since o is a function such that $o(t)/t \to 0$ as $t \to 0$. Then this implies that $\mathcal{L}_{11}(c, \delta) = o(1)$ which is a contradiction to our assumption.

We now show that roots are well-separated in the sense that for any root r there exists a $\epsilon > 0$ such that $|r - r'| > \epsilon$ for any other root r'. We prove this claim by contradiction. I.e., suppose there exists a root, r, such that for any $\epsilon > 0$ there exists another root within ϵ distance of r. This implies the existence of a sequence of roots $(r_n)_n$ that converges to r. For any interval $[r_n, r_{n+1}]$ by Rolle's Theorem there exists a c_n such that $\mathcal{L}_1(c_n, \delta) = 0$. So we constructed a sequence of critical points $(c_n)_n$. Since the sequence of roots converges, this sequence has an accumulation point given by c_∞, which is also a critical point because of continuity of the first derivative of the locator function. But this implies that c_∞ is a critical point that is not isolated, a contradiction.

If, in addition there exists a compact set S such that $|\mathcal{L}(s, \delta)| > 0$ for all $s \notin S$, then we can take S as our effective state space. Since the locator function is assumed to be continuous the set of roots is a closed subset of compact set, thus compact. Each point in this set is isolated, and since a compact set of isolated points is necessarily finite the claim follows. □

SM.5 Proofs of Auxiliary Lemmas

Proof of Lemma C.1. **Parts 1-2.** We only prove the first part because the proof of the second part is completely analogous and thus can be omitted.

Either $\Gamma(\underline{s}, \delta) \ni \underline{s}$ or $\Gamma(\underline{s}, \delta) > \underline{s}$. If the latter holds, there is nothing to prove, so we proceed under the assumption $\Gamma(\underline{s}, \delta) \ni \underline{s}$. Hence, it remains to show that in this case, if $\mathcal{L}(\underline{s}, \delta) > 0$ then there exists a $\varepsilon > 0$ such that $\Gamma(s, \delta) > s$ for all $s \in (\underline{s}, \underline{s} + \varepsilon)$.

We do this by contradiction, i.e., suppose there exists a sequence $(s_n)_n$ converging to $\underline{s}$ such that for each s_n, there exists a $s'_n \in \Gamma(s_n, \delta)$ such that $s'_n \leq s_n$ for all n.

We claim that for each s_n, there exists a $\bar{\Delta} > 0$ such that $\pi(s_n, s'_n) + \delta V(s'_n, \delta) \geq \pi(s_n, s'_n + \Delta) + \delta V(s'_n + \Delta, \delta)$ for any $\Delta \in [0, \bar{\Delta}]$. Since s'_n is optimal for s_n the previous inequality will hold as long as s'_n is not at the "upper boundary" of $Y(s_n)$. Indeed, s'_n is not at the upper bound because if $s'_n = \max Y(s_n)$ then, since $Y^o(s_n) \ni s_n$, it follows that $s'_n > s_n$ a contradiction.

Re-arranging terms and taking limits where Δ converges to zero, it follows that by Proposition 3.1,

$$0 \geq \pi_2(s_n, s'_n) + \delta \max_{y \in \Gamma(s'_n, \delta)} \pi_1(s'_n, y).$$

Since this holds for each n, we can take limits

$$0 \geq \lim_{n \to \infty} \pi_2(s_n, s'_n) + \delta \lim_{n \to \infty} \max_{y \in \Gamma(s'_n, \delta)} \pi_1(s'_n, y).$$

Since $s'_n \leq s_n$ and $s_n \to \underline{s}$, $(s'_n)_n$ converges to $\underline{s}$. This and the fact that π_1 is continuous imply by the ToM that $\lim_{n \to \infty} \max_{y \in \Gamma(s'_n, \delta)} \pi_1(s'_n, y) = \max_{y \in \Gamma(\underline{s}, \delta)} \pi_1(\underline{s}, y)$. So finally, since π_2 is continuous, the previous display implies

$$0 \geq \pi_2(\underline{s}, \underline{s}) + \delta \max_{y \in \Gamma(\underline{s}, \delta)} \pi_1(\underline{s}, y) \geq \mathcal{L}(\underline{s}, \delta).$$

where the last line follows because $\Gamma(\underline{s}, \delta) \ni \underline{s}$. But this violates the assumption that $\mathcal{L}(\underline{s}, \delta) > 0$ and thus arrived at a contradiction.

Parts 3-4. We only prove the third part because the proof of the fourth part is completely analogous and thus can be omitted. We prove the result by contradiction, i.e., suppose there exists a sequence $(s_n)_n$ converging to $\underline{s}$ such that for each s_n, there exists a $s'_n \in \Gamma(s_n, \delta)$ such that $s'_n \geq s_n$ for all n. We claim that for each s_n, there exists a $\bar{\Delta} > 0$ such that $\pi(s_n, s'_n) + \delta V(s'_n, \delta) \geq \pi(s_n, s'_n - \Delta) + \delta V(s'_n - \Delta, \delta)$ for any $\Delta \in [0, \bar{\Delta}]$. The inequality follows from the fact that s'_n is an argmax for s_n provided s'_n is not at the "lower boundary" of $Y(s_n)$. This is true because if $s'_n = \min Y(s_n)$ then, since $Y^o(s_n) \ni s_n$, it follows that $s'_n < s_n$ a contradiction.

Re-arranging terms, dividing by $-\Delta$, and taking limits where Δ converges to zero, it follows that by Proposition 3.1,

$$0 \leq \pi_2(s_n, s'_n) + \delta \min_{y \in \Gamma(s'_n, \delta)} \pi_1(s'_n, y).$$

Since this holds for each n, we can take limits

$$0 \leq \lim_{n \to \infty} \pi_2(s_n, s'_n) + \delta \lim_{n \to \infty} \min_{y \in \Gamma(s'_n, \delta)} \pi_1(s'_n, y).$$

By going to a subsequence if necessary, $(s'_n)_n$ converges to $\underline{s}'$. Since Γ is compact-valued and UHC (see Lemma 3.2), it follows that $\underline{s}' \in \Gamma(\underline{s}, \delta)$, and moreover, by continuity of π_1 and the ToM, $\lim_{n\to\infty} \min_{y \in \Gamma(s'_n, \delta)} \pi_1(s'_n, y) = \min_{y \in \Gamma(\underline{s}', \delta)} \pi_1(\underline{s}', y)$. This, the fact that π_2 is continuous, imply that the previous display equals

$$0 \leq \pi_2(\underline{s}, \underline{s}') + \delta \min_{y \in \Gamma(\underline{s}', \delta)} \pi_1(\underline{s}', y).$$

Finally, since $\Gamma(\underline{s}, \delta) = \underline{s}$ by assumption, $\underline{s} = \underline{s}'$, so

$$0 \leq \pi_2(\underline{s}, \underline{s}) + \delta \min_{y \in \Gamma(\underline{s}, \delta)} \pi_1(\underline{s}, y) = \pi_2(\underline{s}, \underline{s}) + \delta \pi_1(\underline{s}, \underline{s}) = \mathcal{L}(\underline{s}, \delta).$$

But this violates the assumption that $\mathcal{L}(\underline{s}, \delta) < 0$ and we thus arrived at a contradiction.

□

Proof of Lemma C.2. PART 1. By Lemma C.1, for which all assumptions hold, we know that either $\Gamma(\underline{s}, \delta) > \underline{s}$ or $\Gamma(s, \delta) > s$ for all s sufficiently close to $\underline{s}$. Hence, either $\bar{\Gamma}(\cdot, \delta) > 0$ over $\mathbb{S}^o$, and e is not a "true" fixed point and $c = \underline{s}$; or, $\bar{\Gamma}(\cdot, \delta)$ changes signs in $\mathbb{S}^o$. By Lemma SM.3.3 in Appendix SM.3, $\bar{\Gamma}(\cdot, \delta)$ cannot "jump down", so in this case: $\bar{\Gamma}(s, \delta) = 0$ for some s in the interior of $\mathbb{S}$. By Theorem 1 and the uniqueness assumption, such element is e. Moreover, over $[\underline{s}, e)$, $\bar{\Gamma}(\cdot, \delta) > 0$ and $\bar{\Gamma}(e, \delta) = e$. Let $c > e$ be the largest element such that for all $s \in [e, c)$, $\bar{\Gamma}(s, \delta) < 0$. It is clear that $[\underline{s}, c) \subseteq \mathcal{B}(e, \delta)$. Moreover, if $c < \bar{s}$, then c is a Skiba point as otherwise c would be a fixed point but it contradicts the assumption of uniqueness. Finally, for any $s \in (c, \bar{s}]$, $\bar{\Gamma}(s, \delta) > 0$, thus $s \in \mathcal{B}(\bar{s}, \delta)$.

PART 2. By analogous arguments to those presented in Part 1, either $\Gamma(\bar{s},\delta) < \bar{s}$ or $\Gamma(s,\delta) < s$ for all s sufficiently close to $\bar{s}$. So, either e is not a "true" fixed point, and so $c = \bar{s}$; or e is a fixed point of $\Gamma(\cdot,\delta)$. By following an analogous reasoning to that in part 1, we can conclude that there exists a c such that $[\underline{s}, c) \subseteq \mathcal{B}(\underline{s},\delta)$ and $(c,\bar{s}) \subseteq \mathcal{B}(e,\delta)$. □

Proof of Lemma C.3. By Theorem 1 one side of the inclusion holds, so we only show the other — i.e., if $s \in \mathcal{R}^o[\mathcal{L}(\cdot,\delta)]$, then s is a (interior) steady state.

Suppose not. That is, suppose there exists a $s \in \mathbb{S}^o$ such that $\gamma(s) := \Gamma(s,\delta) \neq s$ but $\mathcal{L}(s,\delta) = 0$. We first consider the case $\gamma(s) > s$.

By strict concavity of π it is well-known that $s \mapsto V(s,\delta)$ is also strictly concave. Then by the FOC, which are sufficient,

$$\pi_2(s,\gamma(s)) + \delta\pi_1(\gamma(s),\Gamma(\gamma(s),\delta)) \geq 0$$

(with equality if $\gamma(s) \in \Upsilon^o(s)$). Since, by strict concavity, $s \mapsto V'(s,\delta) := \pi_1(s,\Gamma(s,\delta))$ is strictly decreasing. This fact, the fact that $\gamma(s) > s$, and the previous display thus imply

$$\pi_2(s,\gamma(s)) + \delta\pi_1(s,\gamma(s)) > 0.$$

This result and the condition in the lemma imply $0 < \max_{a\in\Upsilon(s)\,:\,a>s}\{\pi_2(s,a) + \delta\pi_1(s,a)\} \leq \mathcal{L}(s,\delta)$. But this contradicts that $\mathcal{L}(s,\delta) = 0$.

Consider now the case $\gamma(s) < s$. Analogous arguments imply

$$\pi_2(s,\gamma(s)) + \delta\pi_1(s,\gamma(s)) < 0.$$

By the condition in the lemma, $\min_{a\in\Upsilon(s)\,:\,a<s}\{\pi_2(s,a) + \delta\pi_1(s,a)\} \geq \mathcal{L}(s,\delta)$. This implies that $\mathcal{L}(s,\delta) < 0$, a contradiction. □

Proof of Lemma C.4. We first consider the case where e is unstable. By Theorem 1, $\mathcal{L}_1(e,\delta) > 0$. Therefore, $\mathcal{B}[\mathcal{L}(\cdot,\delta)](e) = \{e\}$, and thus there is nothing to prove.

So, henceforth, we take e to be stable. By Proposition SM.3.1, it is sufficient to show

$$\mathcal{B}[\bar{\Gamma}(\cdot,\delta)](e) \supseteq \mathcal{B}[\mathcal{L}(\cdot,\delta)](e).$$

For any $s \in \mathcal{B}[\mathcal{L}(\cdot,\delta)](e)$, either $s < e$ and $\mathcal{L}(s',\delta) > 0$ for all $s' \in (s,e)$, or $s > e$ and $\mathcal{L}(s',\delta) < 0$ for all $s' \in (e,s)$ (if $s = e$ there is nothing to prove). We consider the first case — the second case is omitted as the proof is completely analogous.

Given that $s < e$, to show that $s \in \mathcal{B}[\bar{\Gamma}(\cdot,\delta)](e)$ it suffices to show that $\bar{\Gamma}(s',\delta) > 0$ for any $s' \in (s,e]$. We prove this statement by contradiction; i.e., there exists a $s' \in (s,e]$ such that $\bar{\Gamma}(s',\delta) \leq 0$.

If $\bar{\Gamma}(s',\delta) = 0$, then $s' \in \mathcal{R}^o[\bar{\Gamma}(\cdot,\delta)]$. By theorem 1 this implies that $s' \in \mathcal{R}^o[\mathcal{L}(\cdot,\delta)]$, but this contradicts the fact that $s \in \mathcal{B}[\mathcal{L}(\cdot,\delta)](e)$. Thus $\bar{\Gamma}(s',\delta) < 0$.

Therefore, $\Gamma(s',\delta) < s'$. Under our assumption in the text, e is isolated, thereby implying there exists an open neighborhood of e, $(e-\gamma, e+\gamma)$ for some $\gamma > 0$, such that $s \mapsto \bar{\Gamma}(s,\delta)$ is monotonic. Moreover, since e is taken to be stable, $s \mapsto \bar{\Gamma}(s,\delta)$ is decreasing monotonic, thereby implying $\Gamma(s,\delta) > s$ for all $s \in (e-\gamma, e)$. So, by taken γ sufficiently small, we obtained that $\Gamma(s',\delta) < s'$ and $\Gamma(s'',\delta) > s''$ for (s',s'') such that $s' < e - \gamma < s'' < e$. Since $s \mapsto \bar{\Gamma}(s,\delta)$ is assumed to be a function, it is a continuous one (Lemma SM.3.1 in Appendix SM.3). Thus, by Bolzano, there exists a $c \in (s',s'')$ such that $\Gamma(c,\delta) = c$. But by Theorem 1 this implies that $c \in \mathcal{R}[\mathcal{L}(\cdot,\delta)]$. However, since $s < s'$ and $s'' < e$, we found a point c that $c \in \mathcal{R}[\mathcal{L}(\cdot,\delta)]$ and $c \in (s,e)$. This fact contradicts the assumption that $s \in \mathcal{B}[\mathcal{L}(\cdot,\delta)](e)$.[37] □

SM.6 Sufficient Conditions for Condition 4

Lemma SM.6.1. *Condition 4 in the Proposition 4.3 is implied by either one of the following conditions:*

1. *There exists functions F, G_1, G_2 such that $F > 0$ and $\pi_1(s,s') = F(s,s')G_1(s)$ and $\pi_2(s,s') = F(s,s')G_2(s)$.*

[37]The assumption of $s \mapsto \Gamma(s,\delta)$ being a function is used to obtained this contradiction. Otherwise, the point c could be a Skiba point (a point where $\Gamma(\cdot,\delta)$ "jumps") and will not be “picked up" by $\mathcal{L}$.

2.

$$\sup_{a\in\Upsilon(s):\, a>s} \pi_2(s,a) - \pi_2(s,s) + \delta(\pi_1(s,a) - \pi_1(s,s)) \leq 0, \tag{14}$$

$$\text{and} \inf_{a\in\Upsilon(s):\, a<s} \pi_2(s,a) - \pi_2(s,s) + \delta(\pi_1(s,a) - \pi_1(s,s)) \geq 0 \tag{15}$$

3. $\pi_{22} + \delta\pi_{12} \leq 0$.

Proof. **(1).** By the functional form assumption, $\mathcal{L}(s,\delta) = F(s,s)(G_2(s) + \delta G_1(s))$. Since $F > 0$, $sign\{\mathcal{L}(s,\delta)\} = sign\{G_2(s) + \delta G_1(s)\}$. On the other hand, $sign\,\{\pi_2(s,a) + \delta\pi_1(s,a)\} = sign\,\{F(s,a)(G_2(s) + \delta G_1(s))\} = sign\{G_2(s) + \delta G_1(s)\}$. Thus condition 4 readily holds.

(2). Take any $s \in \mathbb{S}^o$ such that $0 = \mathcal{L}(s,\delta)$. Then, for any $a \in \mathbb{S}$,

$$0 = \mathcal{L}(s,\delta) = \pi_2(s,a) + \delta\pi_1(s,a) - \{\pi_2(s,a) - \pi_2(s,s) + \delta(\pi_1(s,a) - \pi_1(s,s))\}.$$

If $a > s$, then by condition 14, the term in the curly brackets is negative, so $\pi_2(s,a) + \delta\pi_1(s,a) < 0$. Since this holds for any $a > s$, it follows that $\sup_{a\in\Upsilon(s):\, a>s} \pi_2(s,a) + \delta\pi_1(s,a) \leq 0$. If $a < s$, then by condition 15, the term in the curly brackets is positive, so $\pi_2(s,a) + \delta\pi_1(s,a) > 0$. Since this holds for any $a < s$, it follows that $\inf_{a\in\Upsilon(s):\, a<s} \pi_2(s,a) + \delta\pi_1(s,a) \geq 0$. Hence, condition 4 holds.

(3). We show that $\pi_{22} + \delta\pi_{12} \leq 0$ implies the conditions in part 2. Take the case $a > s$. By the MVT, condition 14 can be cast as

$$\sup_{a\in\Upsilon(s):\, a>s} \int_0^1 (\pi_{22}(s, s + t(a-s)) + \delta\pi_{12}(s, s + t(a-s)))dt \times (a-s) \leq 0$$

which is implied by

$$\sup_{a\in\Upsilon(s):\, a>s} \int_0^1 (\pi_{22}(s, s + t(a-s)) + \delta\pi_{12}(s, s + t(a-s)))dt \leq 0.$$

For the second condition in 14 a similar expression holds:

$$\inf_{a \in \Upsilon(s):\, a>s} \int_0^1 (\pi_{22}(s, s+t(a-s)) + \delta\pi_{12}(s, s+t(a-s)))dt \geq 0.$$

A sufficient condition for these inequalities is $(\pi_{22} + \delta\pi_{12})(s, s') \leq 0$ for any $s \in \mathcal{R}^o[\mathcal{L}(\cdot, \delta)]$ and $s' \in \Upsilon(s)$. $\square$

SM.7 Extensions of Application 5

SM.7.1 Import foreign variety

Suppose we allow for foreign variety of good 1, and set the utility from good c_0, domestic c_1 and foreign variety c_1^* is

$$U(c_0, c_1, c_1^*) = u\left(c_1\right) + u^*\left(c_1^*\right) + c_0,$$

and assume that the price of foreign good 1 is fixed at p_1. Assume that this economy is endowed with 1 unit of inelastic labor, so Assumptions 2 and SM.1.1 hold trivially. The next lemma verifies Assumption 1 in this extension.

Lemma SM.7.1. *If $\eta(c_1^*) := -\frac{u^{*'}(c_1^*)}{u^{*''}(c_1^*)c_1^*}$ is always larger than 1, Assumption 1 holds.*

Proof. By Envelope Theorem, we have the same $\pi_1(s, s') = u'(c_1(s, s'))H'(s)F(s')$, which is positive. Besides the good 1 export decision, there is another consumption allocation problem. Given each c_1, s, s':

$$\max_{c_1^*} u^*\left(c_1^*\right) + c_0, \quad s.t. \quad p_1 c_1^* + c_0 = p(e_1)e_1 + G(1-s').$$

Basically, the planner should leverage the revenue from exporting good 1 and (potentially) 0 to finance its consumption of foreign variety of good 1 and good 0. Even though the economy is not necessarily able to achieve an interior solution, our framework can always handle it.

The interior solution of the consumption allocation problem requires $u^{*'}(c_1^*) = p_1$, which means consumption of foreign variety of good 1 is constant. If the export revenue is larger than $p_1 c_1^*$, $p(e_1)e_1 + G(1-s') > p_1 c_1^*$, the first order condition for good 1 export decision, $u'(c_1) = \frac{\varepsilon-1}{\varepsilon} p\left(e_1\right)$, holds and analysis in the Lemma 5.1 remains valid.

If the economy cannot afford to purchase enough foreign variety of good 1 to achieve $u^{*'}(c_1^*) = p_1$, we can approach the monotonic condition differently. We have

$$\pi_{12}(s,s') = u'(c_1(s,s'))H'(s)\left[F'(s') - \frac{1}{\psi(c_1)c_1(s,s')}\frac{\partial c_1(s,s')}{\partial s'}F(s')\right],$$

while we have the first order condition for exporting decision $u'(c_1) = u^{*'}(\frac{p(e_1)e_1+G(1-s')}{p_1})\frac{p(e_1)}{p_1}\left(1-\frac{1}{\varepsilon}\right)$. Fix s and differentiate it, we have

$$\frac{\partial c_1(s,s')}{\partial s'} = \frac{\frac{1-\frac{1}{\varepsilon}}{p_1}\left\{\frac{u^{*''}p^2(e_1)}{p_1}\left[1-\frac{1}{\varepsilon}\right] + u^{*'}p'(e_1)\right\}H(s)F'(s') - \frac{1-\frac{1}{\varepsilon}}{p_1}u^{*''}G'(1-s')\frac{p(e_1)}{p_1}}{u''(c_1) + \frac{1-\frac{1}{\varepsilon}}{p_1}\left\{\frac{u^{*''}p^2(e_1)}{p_1}\left[1-\frac{1}{\varepsilon}\right] + u^{*'}p'(e_1)\right\}}.$$

Substitute it into the π_{12}, we have

$$\pi_{12}(s,s') = u'(c_1)H'(s)F'(s')\left[1 - \frac{F}{\psi c_1}\frac{\frac{1-\frac{1}{\varepsilon}}{p_1}\left\{\frac{u^{*''}p^2(e_1)}{p_1}\left[1-\frac{1}{\varepsilon}\right] + u^{*'}p'(e_1)\right\}H(s) - \frac{1-\frac{1}{\varepsilon}}{p_1}u^{*''}\frac{G'(1-s')}{F'}\frac{p(e_1)}{p_1}}{u''(c_1) + \frac{1-\frac{1}{\varepsilon}}{p_1}\left\{\frac{u^{*''}p^2(e_1)}{p_1}\left[1-\frac{1}{\varepsilon}\right] + u^{*'}p'(e_1)\right\}}\right]$$

Focus on the terms in the bracket,

$$1 - \frac{1}{\psi}\frac{\frac{1-\frac{1}{\varepsilon}}{p_1}\left\{\frac{u^{*''}p^2(e_1)}{p_1}\left[1-\frac{1}{\varepsilon}\right] + u^{*'}p'(e_1)\right\}H(s)F - \frac{1-\frac{1}{\varepsilon}}{p_1}u^{*''}\frac{G'(1-s')F}{F'}\frac{p(e_1)}{p_1}}{u''(c_1)c_1 + \frac{1-\frac{1}{\varepsilon}}{p_1}\left\{\frac{u^{*''}p^2(e_1)}{p_1}\left[1-\frac{1}{\varepsilon}\right] + u^{*'}p'(e_1)\right\}c_1}.$$

Divide u' above and below the fraction line of the second term, and note that $u' = u^{*'}\frac{p(e_1)}{p_1}\left(1-\frac{1}{\varepsilon}\right)$,

$$1 - \frac{1}{\psi}\frac{\left\{\frac{u^{*''}p(e_1)}{u^{*'}p_1}\left[1-\frac{1}{\varepsilon}\right] + \frac{p'(e_1)}{p(e_1)}\right\}H(s)F - \frac{u^{*''}}{u^{*'}}\frac{G'(1-s')F}{F'}\frac{1}{p_1}}{\frac{u''(c_1)c_1}{u'} + \left\{\frac{u^{*''}p(e_1)}{u^{*'}p_1}\left[1-\frac{1}{\varepsilon}\right] + \frac{p'(e_1)}{p(e_1)}\right\}c_1}.$$

Use $e_1 = H(s)F(s') - c_1$ and the definition of ψ, η to get

$$1 - \frac{\left\{-\frac{p(e_1)}{\eta c_1^* p_1}\left[1-\frac{1}{\varepsilon}\right] - \frac{1}{\varepsilon e_1}\right\}(c_1+e_1) + \frac{1}{\eta p_1 c_1^*}\frac{G'(1-s')F}{F'}}{-1 + \left\{-\frac{p(e_1)}{\eta c_1^* p_1}\left[1-\frac{1}{\varepsilon}\right] - \frac{1}{\varepsilon e_1}\right\}\psi c_1}$$

Since we are at a corner in which there is no consumption of good 0 then we have $p_1 c_1^* = pe_1 + G$. Substitute $p_1 c_1^*$ to this term, we have

$$\frac{1 + \left\{\frac{p}{\eta(pe_1+G)}\left[1-\frac{1}{\varepsilon}\right] + \frac{1}{\varepsilon e_1}\right\}\psi c_1 + \left\{-\frac{p}{\eta(pe_1+G)}\left[1-\frac{1}{\varepsilon}\right] - \frac{1}{\varepsilon e_1}\right\}(c_1+e_1) + \frac{1}{\eta(pe_1+G)}\frac{G'(1-s')F}{F'}}{1 + \left\{\frac{p}{\eta(pe_1+G)}\left[1-\frac{1}{\varepsilon}\right] + \frac{1}{\varepsilon e_1}\right\}\psi c_1}$$

Then, we can rearrange terms, and get

$$\frac{(\psi-1)\left\{\frac{p}{\eta(pe_1+G)}\left[1-\frac{1}{\varepsilon}\right] + \frac{1}{\varepsilon e_1}\right\}c_1 + \left(1 - \frac{pe_1}{\eta(pe_1+G)}\right)\left[1-\frac{1}{\varepsilon}\right] + \frac{1}{\eta(pe_1+G)}\frac{G'(1-s')F}{F'}}{1 + \left\{\frac{p}{\eta(pe_1+G)}\left[1-\frac{1}{\varepsilon}\right] + \frac{1}{\varepsilon e_1}\right\}\psi c_1}.$$

We note that if $\psi(c_1) > 1, \varepsilon > 1, \eta(c_1^*) > 1$, this expression is always positive. Henceforth, π_{12} is still positive. The intuition for condition $\eta > 1$ is that we need the utility of foreign variety of good 1 of less curvature, so that more production of good 1 will induce more purchasing of foreign variety rather than too much domestic variety. And then the marginal utility of good 1 will not decrease too fast. □

SM.7.2 Closed economy with CES preference

We allow constant elasticity of substitution between good 0 and 1 in a closed economy. Suppose the economy is endowed with 1 unit of labor, so $\mathbb{S} = [0,1]$. The current utility for the closed economy is

$$U(c_0, c_1) = \frac{\psi}{\psi-1}\left(c_0^{\frac{\sigma-1}{\sigma}} + c_1^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\psi-1}{\psi}\frac{\sigma}{\sigma-1}},$$

so we can rewrite the per-period payoff function as follows

$$\pi(s,s') = \frac{\psi}{\psi-1}\left(\left(G(1-s')\right)^{\frac{\sigma-1}{\sigma}} + \left(H(s)F(s')\right)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\psi-1}{\psi}\frac{\sigma}{\sigma-1}}.$$

Obviously, this setting satisfies Assumptions 2 and SM.1.1, and the following Lemma provides a sufficient condition to ensure Assumption 1 holds.

Lemma SM.7.2. *If $1 < \psi < \sigma$, the closed economy with CES preference satisfies Assumption 1.*

Proof. It follows that $\pi_1(s,s') = \left((G(1-s'))^{\frac{\sigma-1}{\sigma}} + (H(s)F(s'))^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\psi-1}{\psi}\frac{\sigma}{\sigma-1}-1} H(s)^{-\frac{1}{\sigma}}F(s')^{\frac{\sigma-1}{\sigma}}H'(s) > 0$, and

$$\begin{aligned}
&\pi_{12}(s,s') \\
&= \left(c_0^{\frac{\sigma-1}{\sigma}} + c_1^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\psi-1}{\psi}\frac{\sigma}{\sigma-1}-2}(HF)^{-\frac{1}{\sigma}}\left\{(\frac{1}{\sigma}-\frac{1}{\psi})H'F\cdot(H^{\frac{\sigma-1}{\sigma}}F^{\frac{1}{\sigma}}F' - G^{-\frac{1}{\sigma}}G') + \frac{\sigma-1}{\sigma}H'F'\left(c_0^{\frac{\sigma-1}{\sigma}} + c_1^{\frac{\sigma-1}{\sigma}}\right)\right\}
\end{aligned}$$

Next, let's focus on terms within the brace, which can be simplified as

$$(1-\frac{1}{\psi})(HF)^{\frac{\sigma-1}{\sigma}}H'F' + \left(\frac{1}{\psi}-\frac{1}{\sigma}\right)FG^{\frac{-1}{\sigma}}G' + H'F'G^{\frac{\sigma-1}{\sigma}}\frac{\sigma-1}{\sigma}.$$

It is positive when $1 < \psi < \sigma$. □

SM.8 On the role of Assumption 1

This section expands the discussion in Section 6 in the text, providing more detailed and formal statements.

The key result underpinning Theorem 1 is Lemma 3.5, which in turn relies on Lemma SM.3.4 in SM.3. This lemma provides a comparative statics result establishing monotonicity of $\delta \mapsto \Gamma(s,\delta)$ in the sense that $\max\Gamma(s,\delta) \leq \min\Gamma(s,\delta')$ for any $\delta' > \delta$.

By inspection of the proof of Lemma SM.3.4, to obtain this result we invoke the Milgrom-Shannon theorem combined with

$$s' \mapsto \frac{dW(s',\delta)}{d\delta} = V(s',\delta) + \delta\frac{dV(s',\delta)}{d\delta} = V(s',\delta) + \delta\left(V(s'',\delta) + \delta\frac{dV(s'',\delta)}{d\delta}\right), \quad (16)$$

being increasing for some $s'' \in \Gamma(s', \delta)$.

This expression captures the marginal effect of the discount factor on the payoff and consists of two components: (i) a direct effect, reflecting how the discount factor influences the valuation of the "next period" payoff — the term given by $V(s', \delta)$ —, and (ii) an indirect effect, reflecting how the discount factor affects the valuation of future payoffs beyond the next period — the term given by $\delta \frac{dV(s',\delta)}{d\delta}$. To show that the overall expression is increasing in s', it suffices to verify that both components are themselves non-decreasing (with one of them increasing). For the direct effect (i), it suffices to show that $s \mapsto V(s, \delta)$ is increasing, which is what is obtained in Lemmas SM.2.2 and SM.2.3. The indirect effect is more nuanced. To establish the monotonicity of this term we iterate forward the previous expression and obtain $\frac{dV(s',\delta)}{d\delta} = \sum_{t=0}^{\infty} \delta^t V(s^t, \delta)$ for some sequence $(s^t)_t$ such that $s^0 := s'$ and $s^{t+1} \in \Gamma(s^t, \delta)$. We call this sequence an admissible sequence for s'. Thus, to show $s' \mapsto \frac{dV(s',\delta)}{d\delta}$ is non-decreasing it suffices to show that for any $s'_1 > s'_2$, there exists an admissible sequence for s'_1 and for s'_2 respectively such that $\sum_{t=0}^{\infty} \delta^t V(s_1^t, \delta) > \sum_{t=0}^{\infty} \delta^t V(s_2^t, \delta)$. To show this, we use that $s \mapsto \Gamma(s, \delta)$ is non-decreasing — a result established in Lemma SM.3.3 in SM.3. By this property of Γ we can find a $s_1^1 \geq s_2^1$ such that $s_j^1 \in \Gamma(s'_j, \delta)$ for $j = 1, 2$. Iterating forward we construct two admissible sequences, one for s'_1 and one for s'_2, with the property that $s_1^t \geq s_2^t$ for all t. Therefore, by the fact that $s \mapsto V(s, \delta)$ is increasing, the desired result follows.

So, there are two key results to obtain useful comparative statics for the optimal policy correspondence with respect to the discount factor: (increasing) monotonicity of the value function and (increasing) monotonicity of the optimal policy correspondence. The main role of Assumption 1 is precisely to establish these results. Furthermore, the monotonicity restriction in Assumption 1(i) is (only) used to establish the monotonicity of the value function, and monotonicity restriction in Assumption 1(iii) is (only) used to establish the monotonicity of the policy correspondence.

More generally, it should be apparent that in order to ensure useful comparative statics for the optimal policy correspondence with respect to the discount factor both the direct and indirect effects in expression 16 should go in the same direction. In a nutshell, we need $s \mapsto V(s, \delta)$ and $s \mapsto \frac{dV(\Gamma(s,\delta),\delta)}{d\delta}$ to be monotonic in the same direction. Thus,

- A combination of $s \mapsto \Gamma(s, \delta)$ *non-increasing* and $s \mapsto V(s, \delta)$ *non-increasing* will *not* yield useful comparative statics. This follows because in this case the direct effect in expression 16 is negative — a higher s' yields a lower $V(s',\delta)$ — but the indirect effect is positive — a higher s' yields a lower admissible sequence which in turn yields higher $s^t \mapsto \sum_{t=0}^{\infty} \delta^t V(s^t,\delta)$. Thereby implying that both effects go in opposite direction.

- A combination of $s \mapsto \Gamma(s,\delta)$ *non-decreasing* and $s \mapsto V(s,\delta)$ *non-increasing* will yield useful comparative statics. This follows because in this case the direct effect in expression 16 is negative — a higher s' yields a lower $V(s',\delta)$ — and the indirect effect is also negative — a higher s' yields a higher admissible sequence (because $s \mapsto \Gamma(s,\delta)$ is *non-decreasing*) which yields lower $s^t \mapsto \sum_{t=0}^{\infty} \delta^t V(s^t,\delta)$. Thereby implying that both effects go in the same direction.

  Contrary to our current result, this case yields $\delta \mapsto \Gamma(s,\delta)$ is monotonic *non-increasing*.

- A combination of $s \mapsto \Gamma(s,\delta)$ *non-increasing* and $s \mapsto V(s,\delta)$ *non-decreasing* will *not* yield useful comparative statics. This follows because in this case the direct effect in expression 16 is positive — a higher s' yields a higher $V(s',\delta)$ — but the indirect effect is positive — a higher s' yields a lower admissible sequence (because $s \mapsto \Gamma(s,\delta)$ is *non-increasing*) which yields lower $s^t \mapsto \sum_{t=0}^{\infty} \delta^t V(s^t,\delta)$. Thereby implying that both effects go in opposite directions.

The first and third case do not yield useful comparative statics for $\delta \mapsto \Gamma(\cdot,\delta)$ — at least not without further assumptions that can establish which of the effect is stronger.

For the second case, however, our methodology extends to allow for a variant of Assumption 1. In this case $\delta \mapsto \Gamma(s,\delta)$ is monotonic *non-increasing*, so the opposite behavior depicted in Figure 3 holds: When the policy correspondence weakly shifts up with δ, stable fixed points cannot shift *right* as δ increases, while unstable ones cannot shift *left*. This observation gives rise to an analogous version of Lemma 3.5, where the mapping $\delta \mapsto q(\delta)$ that traces the steady states is non-*increasing* for stable steady states and is non-*decreasing* for unstable steady states. By following the same steps of the current proof of the Theorem 1 under this new result, it follows that for any interior steady state s, we have $s \in \mathcal{R}^o[\mathcal{L}]$, with s stable if $\mathcal{L}_1(s,\delta) < 0$ and unstable if $\mathcal{L}_1(s,\delta) > 0$.