Spatial Equilibria: The Case of Two Regions

Online Appendix

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January 16, 2023

1 Additional Results for the Case with $\alpha = 1$ and Costly Trade

In Proposition 4 in the main text we provided sufficient conditions for existence and uniqueness of a regular equilibrium in the case with $\alpha = 1$. These conditions do not cover the case when $\beta \in (0, 1)$, $\gamma \in (0, 1)$, $\mu < 1$, and trade costs are "large". Here we provide characterization of regular equilibria in this case.

Let us define the following constants,

$$\tilde{\phi}_1 \equiv \frac{1-\mu}{1+\mu} \cdot \frac{\zeta\overline{\gamma}\overline{\beta}}{1-\zeta\overline{\gamma}\overline{\beta}} \left(\frac{\gamma}{\overline{\gamma}}\right)^{\frac{1}{1-\mu}} G^{\frac{1}{1-\mu}}, \tag{1}$$

$$\tilde{\phi}_2 \equiv \frac{1-\mu}{1+\mu} \cdot \frac{\zeta\gamma\overline{\beta}}{1-\zeta\gamma\overline{\beta}} \left(\frac{\gamma}{\overline{\gamma}}\right)^{-\frac{1}{1-\mu}} G^{-\frac{1}{1-\mu}}, \tag{2}$$

and

$$\tilde{c} \equiv \frac{1-\zeta\gamma\overline{\beta}}{1-\zeta\overline{\gamma}\overline{\beta}} \left(\frac{\gamma}{\overline{\gamma}}\right)^{\frac{1+\mu}{1-\mu}} G^{\frac{2}{1-\mu}}. \tag{3}$$

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Also, let us define the following condition,

$$\zeta\overline{\gamma}\overline{\beta}\phi_2^{\mu-1} + \left(1-\zeta\overline{\gamma}\overline{\beta}\right)\phi_1\phi_2^{\mu} < G < \left[\zeta\gamma\overline{\beta}\phi_1^{\mu-1} + \left(1-\zeta\gamma\overline{\beta}\right)\phi_1^{\mu}\phi_2\right]^{-1}. \tag{4}$$

Constants $\tilde{\phi}_1$ and $\tilde{\phi}_2$ here have the same definitions as in the main text. Also, condition (4) here is the same as condition (14) in the main text.

Proposition 1 (Regular equilibria under costly trade and $\alpha = 1$). *Assume that $\alpha = 1$, $\beta \in (0,1)$, $\gamma \in (0,1)$, $\mu < 1$, $\phi_1\phi_2 < 1$, $\phi_1 < \tilde{\phi}_1$, and $\phi_2 < \tilde{\phi}_2$, where $\tilde{\phi}_1$ and $\tilde{\phi}_2$ are defined in (1)-(2). Then for any fixed $c = \phi_1/\phi_2$ there exist $\tilde{\tilde{\phi}}_2(c) \in (0, \tilde{\phi}_2]$ and $\overline{\phi}_2(c) \in \left(0, \tilde{\tilde{\phi}}_2(c)\right]$ such that:*

- *(a) If $\phi_2 < \overline{\phi}_2(c)$ then the economy of has a unique regular equilibrium, which is locally stable.*
- *(b) If $\phi_2 > \tilde{\tilde{\phi}}_2(c)$ and condition (4) holds then the economy has a unique regular equilibrium, which is unstable. If $\phi_2 > \tilde{\tilde{\phi}}_2(c)$ but condition (4) does not hold, then the economy does not have regular equilibria.*
- *(c) If $\overline{\phi}_2(c) \leq \phi_2 < \tilde{\tilde{\phi}}_2(c)$ then the economy has at most three regular equilibria. Two of these equilibria are unstable and one is locally stable.*
- *(d) If $\phi_2 = \tilde{\tilde{\phi}}_2(c)$ then:*
  - *If $c \neq \tilde{c}$ — with $\tilde{c}$ given by (3) — then the economy has at most two regular equilibria, one is unstable and one is neither stable nor unstable;*
  - *If $c = \tilde{c}$ then $\tilde{\tilde{\phi}}_2(\tilde{c}) = \tilde{\phi}_2$ and the economy has at most one regular equilibrium, which is neither stable nor unstable.*

In the proof of Proposition 1, we use the notation introduced in the proof of Proposition 4 in the main text. In particular, in order to characterize the number of regular equilibria of the economy, we work with the following equation,

$$\tilde{V}(z) \equiv \zeta\overline{\gamma}\overline{\beta}z^{\frac{1}{1+\mu}} - \zeta\gamma\overline{\beta}Gz^{\frac{\mu}{1+\mu}} - \left(1-\zeta\gamma\overline{\beta}\right)\phi_2 Gz + \left(1-\zeta\overline{\gamma}\overline{\beta}\right)\phi_1. \tag{5}$$

Recall that any positive solution to this equation will correspond to a regular equilibrium of the economy only if it falls within the interval $\left(\phi_1^{1+\mu}, \phi_2^{-(1+\mu)}\right)$.

As we show below, the two-dimensional space (ϕ_1, ϕ_2) can be divided into two areas: one area where equation (5) has at most one positive solution and one area where equation (5) has at most three positive solutions. A natural and insightful way to describe these areas is to use parameters ϕ_2 and $c \equiv \phi_1/\phi_2$ (rather than ϕ_1 and ϕ_2) to trace the "uniqueness boundary" that separates these areas. A benefit of this approach is that it naturally covers the case of symmetric trade costs with $c = 1$.

The main result that helps to prove Proposition 1 is the following lemma.

Lemma 1. *Assume that that $\alpha = 1$, $\beta \in (0,1)$, $\gamma \in (0,1)$, $\mu < 1$, $\phi_1\phi_2 < 1$, $\phi_1 < \tilde{\phi}_1$, and $\phi_2 < \tilde{\phi}_2$. Then for any fixed $c = \phi_1/\phi_2$ there exists $\tilde{\tilde{\phi}}_2(c) \in (0, \tilde{\phi}_2]$ such that for any $\phi_2 > \tilde{\tilde{\phi}}_2(c)$ equation $\tilde{V}(z) = 0$ has one positive solution, and for any $\phi_2 < \tilde{\tilde{\phi}}_2(c)$ equation $\tilde{V}(z) = 0$ has three positive solutions. For $\phi_2 = \tilde{\tilde{\phi}}_2(c)$, equation $\tilde{V}(z) = 0$ generally has two positive solutions except for the special case with $c = \tilde{c}$ given by (3), in which case $\tilde{\tilde{\phi}}_2(\tilde{c}) = \tilde{\phi}_2$ and equation $\tilde{V}(z) = 0$ has one positive solution.*

Proof. Replace ϕ_1 in the definition of $\tilde{V}$ by $c\phi_2$ and introduce arguments ϕ_2 and c into the notation of $\tilde{V}$ by writing this function as

$$\tilde{V}(z, \phi_2, c) \equiv \zeta\overline{\gamma}\overline{\beta} z^{\frac{1}{1+\mu}} - \zeta\gamma\overline{\beta} G z^{\frac{\mu}{1+\mu}} - \left(1 - \zeta\gamma\overline{\beta}\right)\phi_2 G z + \left(1 - \zeta\overline{\gamma}\overline{\beta}\right) c\phi_2.$$

Denote the first and the second derivatives of $\tilde{V}(z, \phi_2, c)$ with respect to z as $\tilde{V}_1'(z, \phi_2, c)$ and $\tilde{V}_1''(z, \phi_2, c)$. These derivatives are given by

$$\tilde{V}_1'(z, \phi_2, c) = \frac{1}{1+\mu}\zeta\overline{\gamma}\overline{\beta} z^{-\frac{\mu}{1+\mu}} - \frac{\mu}{1+\mu}\zeta\gamma\overline{\beta} G z^{-\frac{1}{1+\mu}} - \left(1 - \zeta\gamma\overline{\beta}\right)\phi_2 G, \tag{6}$$

$$\tilde{V}_1''(z, \phi_2, c) = \frac{\mu\zeta\gamma\overline{\beta} G}{(1+\mu)^2} z^{-\frac{\mu}{1+\mu}-1}\left(z^{-\frac{1-\mu}{1+\mu}} - \overline{z}_0^{-\frac{1-\mu}{1+\mu}}\right), \tag{7}$$

where

$$\overline{z}_0 \equiv \left(\frac{\gamma}{\overline{\gamma}} G\right)^{\frac{1+\mu}{1-\mu}}. \tag{8}$$

We break the proof of the current lemma into two steps.

Step 1. We first prove that for each $c > 0$ the system of equations $\tilde{V}(z, \phi_2, c) = 0$ and $\tilde{V}_1'(z, \phi_2, c) = 0$ has a unique solution $\tilde{\tilde{z}}(c)$ and $\tilde{\tilde{\phi}}_2(c)$ with $\tilde{\tilde{z}}(c) > 0$ and $0 < \tilde{\tilde{\phi}}_2(c) \leq \tilde{\phi}_2$.

Solving for ϕ_2 from equation $\tilde{V}_1'(z, \phi_2, c) = 0$, substituting the result into equation $\tilde{V}(z, \phi_2, c) = 0$, and after doing some algebra, we get equation $H(z, c) = 0$ in z, where

$$H(z, c) \equiv \mu\overline{\gamma} z^{\frac{2}{1+\mu}} + \frac{1 - \zeta\overline{\gamma}\overline{\beta}}{1 - \zeta\gamma\overline{\beta}}\overline{\gamma} G^{-1} c z^{\frac{1-\mu}{1+\mu}} - \gamma G z - \frac{1 - \zeta\overline{\gamma}\overline{\beta}}{1 - \zeta\gamma\overline{\beta}}\mu\gamma c.$$

We are going to show that for any $c > 0$ there is a unique $\tilde{\tilde{z}}(c) > 0$ such that $H\left(\tilde{\tilde{z}}(c), c\right) = 0$.

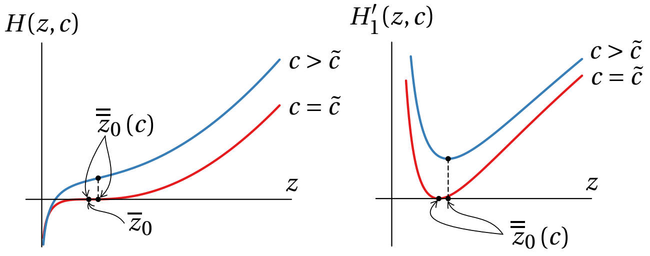


Figure 1: Krugman case, $\mu < 1$: H and H'

The first and the second derivatives of $H\left(z,c\right)$ with respect to z are given by

$$H_1'\left(z,c\right) = \frac{2\mu}{1+\mu}\overline{\gamma} z^{\frac{1-\mu}{1+\mu}} + \frac{1-\mu}{1+\mu} \cdot \frac{1-\zeta\overline{\gamma}\overline{\beta}}{1-\zeta\gamma\overline{\beta}}\overline{\gamma} G^{-1} c z^{-\frac{2\mu}{1+\mu}} - \gamma G,$$
$$H_1''\left(z,c\right) = \frac{2\mu\left(1-\mu\right)}{\left(1+\mu\right)^2}\overline{\gamma} z^{-\frac{2\mu}{1+\mu}-1}\left(z - \overline{\overline{z}}_0\left(c\right)\right),$$

where

$$\overline{\overline{z}}_0\left(c\right) \equiv \frac{1-\zeta\overline{\gamma}\overline{\beta}}{1-\zeta\gamma\overline{\beta}} G^{-1} c. \tag{9}$$

From here we see that $H_1''\left(z,c\right) < 0$ if and only if $z < \overline{\overline{z}}_0\left(c\right)$. Thus, for any $c > 0$, $H_1'\left(z,c\right)$ is a convex function of z, and $\overline{\overline{z}}_0\left(c\right)$ is its minimum (see Figure 1, which sketches the shapes of $H\left(z,c\right)$ and $H_1'\left(z,c\right)$).

Evaluating $H_1'\left(\overline{\overline{z}}_0\left(c\right),c\right)$, we get

$$H_1'\left(\overline{\overline{z}}_0\left(c\right),c\right) = \overline{\gamma}\left(\frac{1-\zeta\overline{\gamma}\overline{\beta}}{1-\zeta\gamma\overline{\beta}} G^{-1} c\right)^{\frac{1-\mu}{1+\mu}} - \gamma G,$$

which allows us to see that $H_1'\left(\overline{\overline{z}}_0\left(c\right),c\right) > 0$ if and only if $c > \tilde{c}$, where $\tilde{c}$ was defined in (3). From here we immediately see that if $c > \tilde{c}$, then $H_1'\left(z,c\right) > 0$ for all z, and so $H\left(z,c\right)$ increases in z. Then, given that $H\left(0,c\right) < 0$ and $\lim_{z\to\infty} H\left(z,c\right) = \infty$, we conclude that, for any fixed $c > \tilde{c}$, $H\left(z,c\right)$ intersects the horizontal axis $z = 0$ only once at some $\tilde{\tilde{z}}\left(c\right) > 0$.

The case with $c = \tilde{c}$ is similar to the case with $c > \tilde{c}$ with the only difference that $H\left(z,c\right)$ is an increasing function for all $z \neq \overline{\overline{z}}_0\left(\tilde{c}\right)$ and $H\left(\overline{\overline{z}}_0\left(\tilde{c}\right),\tilde{c}\right) = H_1'\left(\overline{\overline{z}}_0\left(\tilde{c}\right),\tilde{c}\right) = 0$ (and so $\tilde{\tilde{z}}\left(\tilde{c}\right) = \overline{\overline{z}}_0\left(\tilde{c}\right)$).

Now consider the case with $c < \tilde{c}$. Let

$$\begin{aligned}\tilde{H}\left(z,c\right) &\equiv -\frac{1-\zeta\gamma\overline{\beta}}{1-\zeta\overline{\gamma}\overline{\overline{\beta}}}c^{-1}z^{\frac{2}{1+\mu}}H\left(z^{-1},c\right) \\ &= \mu\gamma z^{\frac{2}{1+\mu}} + \frac{1-\zeta\gamma\overline{\beta}}{1-\zeta\overline{\gamma}\overline{\overline{\beta}}}\gamma G c^{-1} z^{\frac{1-\mu}{1+\mu}} - \overline{\gamma}G^{-1}z - \frac{1-\zeta\gamma\overline{\beta}}{1-\zeta\overline{\gamma}\overline{\overline{\beta}}}\mu\overline{\gamma}c^{-1}.\end{aligned}$$

Obviously, functions $H\left(z,c\right)$ and $\tilde{H}\left(z,c\right)$ have the same number of zeros for $z > 0$. Observe that function $\tilde{H}\left(z,c\right)$ is similar to function $H\left(z,c\right)$ with the difference that γ is swapped with $\overline{\gamma}$; G is swapped with G^{-1}; and c is swapped with c^{-1}. Applying the same analysis to function $\tilde{H}\left(z,c\right)$ as to function $H\left(z,c\right)$, we get that $\tilde{H}\left(z,c\right)$ increases in z if $c < \tilde{c}$. Then, given that $\tilde{H}\left(0,c\right) < 0$ and $\lim_{z\to\infty}\tilde{H}\left(z,c\right) = \infty$, we conclude that $\tilde{H}\left(z,c\right)$ intersects the horizontal axis $z = 0$ once and only once for some $\tilde{\tilde{z}}\left(c\right) > 0$. The corresponding unique solution to equation $H\left(z,c\right) = 0$ is $[\tilde{\tilde{z}}\left(c\right)]^{-1}$.

At this point, we have established that for any $c > 0$ there is a unique $\tilde{\tilde{z}}\left(c\right) > 0$ such that $H\left(\tilde{\tilde{z}}\left(c\right),c\right) = 0$. This, of course, means that our original system $\tilde{V}\left(z,\phi_2,c\right) = 0$ and $\tilde{V}_1'\left(z,\phi_2,c\right) = 0$ has a unique solution $\left(\tilde{\tilde{z}}\left(c\right),\tilde{\tilde{\phi}}_2\left(c\right)\right)$ with $\tilde{\tilde{z}}\left(c\right) > 0$, where we use equation $\tilde{V}_1'\left(z,\phi_2,c\right) = 0$ to find $\tilde{\tilde{\phi}}_2\left(c\right)$ corresponding to $\tilde{\tilde{z}}\left(c\right)$, which gives

$$\tilde{\tilde{\phi}}_2\left(c\right) = \frac{1}{1+\mu}\cdot\frac{\zeta\overline{\gamma}\overline{\beta}}{1-\zeta\gamma\overline{\overline{\beta}}}G^{-1}\left[\tilde{\tilde{z}}\left(c\right)\right]^{-\frac{\mu}{1+\mu}} - \frac{\mu}{1+\mu}\cdot\frac{\zeta\gamma\overline{\beta}}{1-\zeta\gamma\overline{\overline{\beta}}}\left[\tilde{\tilde{z}}\left(c\right)\right]^{-\frac{1}{1+\mu}}.$$

We need to verify that $0 < \tilde{\tilde{\phi}}_2\left(c\right) \leq \tilde{\phi}_2$. The upper bound $\tilde{\tilde{\phi}}_2\left(c\right) \leq \tilde{\phi}_2$ simply follows from the fact that $\tilde{V}_1'\left(\tilde{\tilde{z}}\left(c\right),\tilde{\tilde{\phi}}_2\left(c\right),c\right) = 0$, and we know from the proof of Lemma 3 in Appendix B.6 of the main text that $\tilde{V}_1'\left(z,\phi_2,c\right) < 0$ for all $z > 0$ and $\phi_2 > \tilde{\phi}_2$. Verifying positivity of $\tilde{\tilde{\phi}}_2\left(c\right)$ is equivalent to verifying that $\tilde{\tilde{z}}\left(c\right) > \left(\mu G\gamma/\overline{\gamma}\right)^{\frac{1+\mu}{1-\mu}}$. We have $H\left(0,c\right) < 0$ and $\lim_{z\to\infty}H\left(z,c\right) = \infty$ and $\tilde{\tilde{z}}\left(c\right) > 0$ is a unique solution to $H\left(z,c\right) = 0$. Therefore, $H\left(z,c\right) < 0$ for $z < \tilde{\tilde{z}}\left(c\right)$ and $H\left(z,c\right) > 0$ for $z > \tilde{\tilde{z}}\left(c\right)$. Simple algebra reveals that

$$H\left(\left(\mu G\gamma/\overline{\gamma}\right)^{\frac{1+\mu}{1-\mu}},c\right) = -\left(\mu^{-2}-1\right)\mu^{1+\frac{2}{1-\mu}}\gamma^{\frac{2}{1-\mu}}\overline{\gamma}^{-\frac{1+\mu}{1-\mu}}G^{\frac{2}{1-\mu}} < 0.$$

This allows us to conclude that $\left(\mu G\gamma/\overline{\gamma}\right)^{\frac{1+\mu}{1-\mu}} < \tilde{\tilde{z}}\left(c\right)$ and, thus, $\tilde{\tilde{\phi}}_2\left(c\right) > 0$.

STEP 2. We are now going to prove the statement of the current lemma about the existence of a unique $\tilde{\tilde{\phi}}_2\left(c\right) > 0$ that traces the uniqueness boundary.

Consider $\tilde{V}_1'\left(z,\phi_2,c\right)$ for any $c > 0$ and $\phi_2 < \tilde{\phi}_2$. This case is illustrated in Figure 2 by the curve labeled "$\phi_2 < \tilde{\phi}_2$". In Lemma 3 in Appendix B.6 of the main text we have

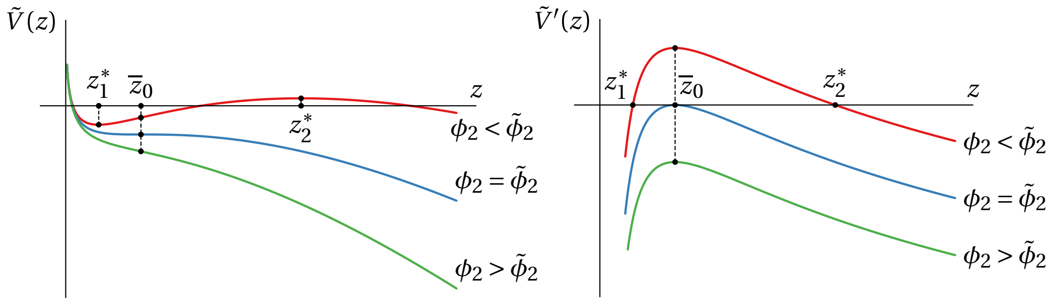


Figure 2: Krugman case, $\mu < 1$: $\tilde{V}$ and $\tilde{V}'$

established that $\tilde{V}_1'(\bar{z}_0, \phi_2, c) > 0$ for $\phi_2 < \tilde{\phi}_2$, where $\bar{z}_0 \equiv (\gamma G/\overline{\gamma})^{\frac{1+\mu}{1-\mu}}$ was defined in (8) and is the global maximum of $\tilde{V}_1'(z, \phi_2, c)$. This implies that $\tilde{V}_1'(z, \phi_2, c)$ intersects the horizontal axis $z = 0$ at exactly two points: one lower than $\bar{z}_0$ and one larger than $\bar{z}_0$. This fact allows us to define functions

$$z_1^*(\phi_2, c) \equiv \left\{ z \middle| z \leq \bar{z}_0 \text{ and } \tilde{V}_1'(z, \phi_2, c) = 0 \right\},$$
$$z_2^*(\phi_2, c) \equiv \left\{ z \middle| z \geq \bar{z}_0 \text{ and } \tilde{V}_1'(z, \phi_2, c) = 0 \right\},$$

both with domain $\phi_2 < \tilde{\phi}_2$ and $c > 0$. Definitions of $z_1^*(\phi_2, c)$ and $z_2^*(\phi_2, c)$ imply that $z_1^*(\phi_2, c) < \bar{z}_0 < z_2^*(\phi_2, c)$. Moreover, we have that $\tilde{V}_1'(z, \phi_2, c) < 0$ for $z \in (0, z_1^*(\phi_2, c)) \cup (z_2^*(\phi_2, c), \infty)$ and $V_1'(z, \phi_2, c) > 0$ for $z \in (z_1^*(\phi_2, c), z_2^*(\phi_2, c))$. Therefore, $z_1^*(\phi_2, c)$ is a local minimum of $\tilde{V}(z, \phi_2, c)$ and $z_2^*(\phi_2, c)$ is a local maximum of $\tilde{V}(z, \phi_2, c)$, and $\tilde{V}(z_1^*(\phi_2, c), \phi_2, c) < \tilde{V}(z_2^*(\phi_2, c), \phi_2, c)$.

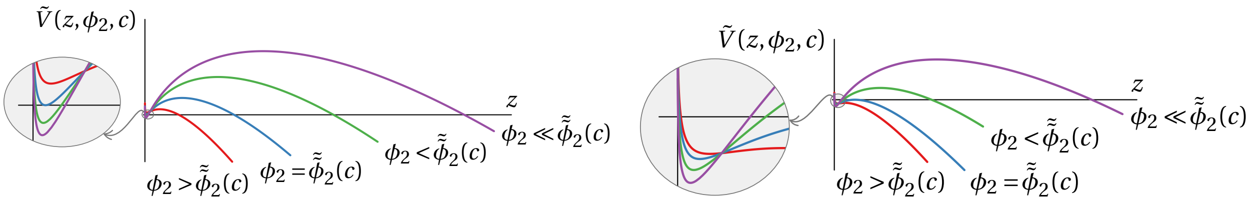
We have shown in Step 1 that there exists a unique solution $\left(\tilde{\tilde{z}}(c), \tilde{\tilde{\phi}}_2(c)\right)$ to the system of equations $\tilde{V}(z, \phi_2, c) = 0$ and $\tilde{V}_1'(z, \phi_2, c) = 0$. Moreover, as we have argued in Step 1, $H(z, c) < 0$ if and only if $z < \tilde{\tilde{z}}(c)$. Simple algebra reveals that

$$H(\bar{z}_0, c) = (1 - \mu)\gamma \frac{1 - \zeta\overline{\gamma}\overline{\beta}}{1 - \zeta\gamma\overline{\beta}}(c - \tilde{c}),$$

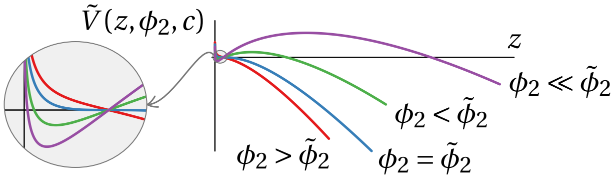
where $\tilde{c} \equiv \frac{1-\zeta\gamma\overline{\beta}}{1-\zeta\overline{\gamma}\overline{\beta}}(\gamma/\overline{\gamma})^{\frac{1+\mu}{1-\mu}} G^{\frac{2}{1-\mu}}$ was defined in (3). Thus, $H(\bar{z}_0, c) < 0$ if and only if $c < \tilde{c}$. Therefore, $\tilde{\tilde{z}}(c) > \bar{z}_0$ if and only if $c < \tilde{c}$. This, in turn, implies that $\tilde{\tilde{z}}(c) = z_1^*\left(\tilde{\tilde{\phi}}_2(c), c\right)$ for $c > \tilde{c}$ and $\tilde{\tilde{z}}(c) = z_2^*\left(\tilde{\tilde{\phi}}_2(c), c\right)$ for $c < \tilde{c}$, while $\tilde{\tilde{z}}(\tilde{c}) = \bar{z}_0$ and $\tilde{\tilde{\phi}}_2(\tilde{c}) = \tilde{\phi}_2$.

Next, using the fact that $\tilde{V}_1'(z_i^*(\phi_2, c), \phi_2, c) = 0$, we find that

$$\frac{d\tilde{V}(z_i^*(\phi_2, c), \phi_2, c)}{d\phi_2} = -(1 - \zeta\gamma\overline{\beta})Gz_i^*(\phi_2, c) + (1 - \zeta\overline{\gamma}\overline{\beta})c,$$



(a) $c > \tilde{c}$ (b) $c < \tilde{c}$



(c) $c = \tilde{c}$

Figure 3: Krugman case, $\mu < 1$: $\tilde{V}$ for different c

and, thus, $d\tilde{V}\left(z_i^*\left(\phi_2, c\right), \phi_2, c\right)/d\phi_2 > 0$ if and only if $z_i^*\left(\phi_2, c\right) < \bar{\bar{z}}_0\left(c\right)$, where $\bar{\bar{z}}_0\left(c\right)$ was defined in (9). Using the definitions of $\bar{z}_0$ and $\bar{\bar{z}}_0\left(c\right)$, we get that $\bar{z}_0 > \bar{\bar{z}}_0\left(c\right)$ if and only if $c < \tilde{c}$. Thus, given the fact that $z_1^*\left(\phi_2, c\right) < \bar{z}_0 < z_2^*\left(\phi_2, c\right)$ for any $\phi_2 < \tilde{\phi}_2$ and $c > 0$, we get that if $c > \tilde{c}$ then $z_1^*\left(\phi_2, c\right) < \bar{z}_0 < \bar{\bar{z}}_0\left(c\right)$, and if $c < \tilde{c}$ then $z_2^*\left(\phi_2, c\right) > \bar{z}_0 > \bar{\bar{z}}_0\left(c\right)$. This, in turn, implies that if $c > \tilde{c}$ then $\tilde{V}\left(z_1^*\left(\phi_2, c\right), \phi_2, c\right)$ is increasing in ϕ_2, and if $c < \tilde{c}$ then $\tilde{V}\left(z_2^*\left(\phi_2, c\right), \phi_2, c\right)$ is decreasing in ϕ_2.

We are now ready to bring all facts together to characterize multiplicity of solutions of equation $\tilde{V}\left(z, \phi_2, c\right) = 0$. Fix any $c > \tilde{c}$ and consider function $\tilde{V}\left(z, \phi_2, c\right)$ as we change ϕ_2 (see Figure 3a). For $\phi_2 = \tilde{\tilde{\phi}}_2\left(c\right)$, the horizontal axis $z = 0$ is tangent to the local minimum of $\tilde{V}\left(z, \phi_2, c\right)$ at point $\tilde{\tilde{z}}\left(c\right) = z_1^*\left(\tilde{\tilde{\phi}}_2\left(c\right), c\right)$ (this case is depicted in Figure 3a by the curve labeled "$\phi_2 = \tilde{\tilde{\phi}}_2\left(c\right)$"). Thus, $\tilde{V}\left(z_1^*\left(\tilde{\tilde{\phi}}_2\left(c\right), c\right), \tilde{\tilde{\phi}}_2\left(c\right), c\right) = 0$ and for all points $z \in \left(0, z_2^*\left(\tilde{\tilde{\phi}}_2\left(c\right), c\right)\right)$ different from $z_1^*\left(\tilde{\tilde{\phi}}_2\left(c\right), c\right)$ we have $\tilde{V}\left(z, \tilde{\tilde{\phi}}_2\left(c\right), c\right) > 0$. For $z > z_2^*\left(\tilde{\tilde{\phi}}_2\left(c\right), c\right)$, function $\tilde{V}\left(z, \tilde{\tilde{\phi}}_2\left(c\right), c\right)$ monotonically decreases from a positive value to $-\infty$ as $z \to \infty$. This implies that for $z > z_2^*\left(\tilde{\tilde{\phi}}_2\left(c\right), c\right)$ function $\tilde{V}\left(z, \tilde{\tilde{\phi}}_2\left(c\right), c\right)$ crosses the horizontal axis $z = 0$ only once at some $\tilde{z}$. Thus, for $\phi_2 = \tilde{\tilde{\phi}}_2\left(c\right)$ there are two solutions to equation $V\left(z, \phi_2, c\right) = 0$: $z_1^*\left(\tilde{\tilde{\phi}}_2\left(c\right), c\right)$ and $\tilde{z}$.

Next, as we have argued above, $\tilde{V}\left(z_1^*\left(\phi_2, c\right), \phi_2, c\right)$ is increasing in ϕ_2 for $c > \tilde{c}$. Therefore, for $\phi_2 > \tilde{\tilde{\phi}}_2\left(c\right)$ we have $\tilde{V}\left(z_1^*\left(\phi_2, c\right), \phi_2, c\right) > 0$, which implies that $\tilde{V}\left(z, \phi_2, c\right) > 0$ for all $z \in \left(0, z_2^*\left(\phi_2, c\right)\right)$ (this case is depicted in Figure 3a by the curve labeled "$\phi_2 > \tilde{\tilde{\phi}}_2\left(c\right)$"). And for $z > z_2^*\left(\phi_2, c\right)$, again, function $\tilde{V}\left(z, \phi_2, c\right)$ intersects the horizontal axis $z = 0$ once and only once. Thus, for $\phi_2 > \tilde{\tilde{\phi}}_2\left(c\right)$ there is a unique solution to equation $\tilde{V}\left(z, \phi_2, c\right) = 0$.

Finally, for $\phi_2 < \tilde{\tilde{\phi}}_2\left(c\right)$ we have $\tilde{V}\left(z_1^*\left(\phi_2, c\right), \phi_2, c\right) < 0$ (this case is depicted in Figure 3a by the curves labeled "$\phi_2 < \tilde{\tilde{\phi}}_2(c)$" and "$\phi_2 \ll \tilde{\tilde{\phi}}_2(c)$" with the latter curve corresponding to a lower value of ϕ_2 than the former curve). At the same time, we necessarily have $\tilde{V}(z_2^*(\phi_2, c), \phi_2, c) > 0$ for $\phi_2 < \tilde{\tilde{\phi}}_2(c)$. To see this, observe that $\tilde{V}(z_2^*(\phi_2, c), \phi_2, c) > 0$ for $\phi_2 \in \left[\tilde{\tilde{\phi}}_2(c), \tilde{\phi}_2\right)$, because $\tilde{V}(z_1^*(\phi_2, c), \phi_2, c) \geq 0$ for $\phi_2 \in \left[\tilde{\tilde{\phi}}_2(c), \tilde{\phi}_2\right)$ and $\tilde{V}(z_1^*(\phi_2, c), \phi_2, c) < \tilde{V}(z_2^*(\phi_2, c), \phi_2, c)$ for all $\phi_2 < \tilde{\phi}_2$. If there is some $\phi_2' \in \left(0, \tilde{\tilde{\phi}}_2(c)\right)$ such that $\tilde{V}(z_2^*(\phi_2', c), \phi_2, c) \leq 0$ then continuity of $\tilde{V}(z_2^*(\phi_2, c), \phi_2, c)$ in ϕ_2 implies that there is also some $\phi_2'' \in \left[\phi_2', \tilde{\tilde{\phi}}_2(c)\right)$ such that $\tilde{V}(z_2^*(\phi_2'', c), \phi_2'', c) = 0$. Then the pair $z_2^*(\phi_2'', c)$ and ϕ_2'' is a solution of the system of equations $\tilde{V}(z, \phi_2, c) = 0$ and $\tilde{V}_1'(z, \phi_2, c) = 0$. Moreover, $\phi_2'' \neq \tilde{\tilde{\phi}}_2(c)$, while the pair $z_1^*\left(\tilde{\tilde{\phi}}_2(c), c\right)$ and $\tilde{\tilde{\phi}}_2(c)$ is another solution of the same system of equations. This contradicts to the fact established at Step 1 that the system of equations $\tilde{V}(z, \phi_2, c) = 0$ and $\tilde{V}_1'(z, \phi_2, c) = 0$ has a unique solution. Thus, indeed, $\tilde{V}(z_2^*(\phi_2, c), \phi_2, c) > 0$ for all $\phi_2 < \tilde{\tilde{\phi}}_2(c)$.

The facts that $\tilde{V}(0, \phi_2, c) > 0$ and that for any $\phi_2 < \tilde{\tilde{\phi}}_2(c)$ we have $\tilde{V}(z_1^*(\phi_2, c), \phi_2, c) < 0$ and $\tilde{V}(z_2^*(\phi_2, c), \phi_2, c) > 0$ imply that for $z \in (0, z_2^*(\phi_2, c))$ function $\tilde{V}(z, \phi_2, c)$ intersects the horizontal axis $z = 0$ exactly two times. In addition to that — similarly to the cases with $\phi_2 > \tilde{\tilde{\phi}}_2(c)$ and $\phi_2 = \tilde{\tilde{\phi}}_2(c)$ — function $\tilde{V}(z, \phi_2, c)$ intersects the horizontal axis $z = 0$ one more time for some $\tilde{z} > z_2^*(\phi_2, c)$. Thus, for $\phi_2 < \tilde{\tilde{\phi}}_2(c)$ equation $\tilde{V}(z, \phi_2, c) = 0$ has three solutions in z.

Analysis of multiplicity of solutions of $\tilde{V}(z, \phi_2, c)$ for $c < \tilde{c}$ is similar to the above analysis with $c > \tilde{c}$ (see Figure 3b for illustration). The difference is that for $c < \tilde{c}$ the horizontal axis $z = 0$ is tangent to the local maximum of $\tilde{V}\left(z, \tilde{\tilde{\phi}}_2(c), c\right)$ at point $\tilde{\tilde{z}}(c) = z_2^*\left(\tilde{\tilde{\phi}}_2(c), c\right)$, and $\tilde{V}(z_2^*(\phi_2, c), \phi_2, c)$ is a decreasing function of ϕ_2.

The case with $c = \tilde{c}$ is special (see Figure 3c for illustration). In this case, $\tilde{V}(\bar{z}_0, \phi_2, \tilde{c}) = 0$ for any ϕ_2. We know from Lemma 3 in Appendix B.6 of the main text that for any $\phi_2 \geq \tilde{\phi}_2$ equation $\tilde{V}(z, \phi_2, c) = 0$ has a unique solution in z. Thus, for all $\phi_2 \geq \tilde{\phi}_2$ the unique solution to $\tilde{V}(z, \phi_2, \tilde{c}) = 0$ is $\bar{z}_0$. For $\phi_2 < \tilde{\phi}_2$, we have that $z_1^*(\phi_2, \tilde{c}) < \bar{z}_0 < z_2^*(\phi_2, \tilde{c})$ and that $\tilde{V}_1'(z, \phi_2, \tilde{c}) < 0$ for $z \in (0, z_1^*(\phi_2, \tilde{c})) \cup (z_2^*(\phi_2, \tilde{c}), \infty)$ and $\tilde{V}_1'(z, \phi_2, \tilde{c}) > 0$ for $z \in (z_1^*(\phi_2, \tilde{c}), z_2^*(\phi_2, \tilde{c}))$. Therefore, $\tilde{V}(z_1^*(\phi_2, \tilde{c}), \phi_2, \tilde{c}) < 0 < \tilde{V}(z_2^*(\phi_2, \tilde{c}), \phi_2, \tilde{c})$. Then, given that $\tilde{V}(0, \phi_2, \tilde{c}) > 0$ and $\lim_{z \to \infty} \tilde{V}(z, \phi_2, \tilde{c}) = -\infty$, we conclude that $\tilde{V}(z, \phi_2, \tilde{c})$ intersects the horizontal axis $z = 0$ once for $z < z_1^*(\phi_2, \tilde{c})$ and once for $z > z_2^*(\phi_2, \tilde{c})$. Thus, overall, for $\phi_2 < \tilde{\phi}_2$, equation $\tilde{V}(z, \phi_2, \tilde{c}) = 0$ has three solutions (one of which is $\bar{z}_0$). $\square$

The next lemma deals with the issue of existence of a regular equilibrium of the economy.

Lemma 2. *Assume that that $\alpha = 1$, $\beta \in (0,1)$, $\gamma \in (0,1)$, $\mu < 1$, $\phi_1\phi_2 < 1$, $\phi_1 < \tilde{\phi}_1$, and $\phi_2 < \tilde{\phi}_2$.*

(i) In any of the cases of Lemma 1, in which equation $\tilde{V}(z) = 0$ has a unique solution, the economy has a regular equilibrium only if condition (4) holds.

(ii) For any fixed $c = \phi_1/\phi_2$ there exists $\overline{\phi}_2(c) \in \left(0, \tilde{\tilde{\phi}}_2(c)\right]$ — with $\tilde{\tilde{\phi}}_2(c)$ defined in Lemma 1 — such that for all $\phi_2 < \overline{\phi}_2(c)$ the economy has a unique regular equilibrium.

Proof. In the proof of this proposition we are going to use notation introduced in Lemma 1.

Part (i). As we have explained before, any solution $\tilde{z}$ to equation $\tilde{V}(z) = 0$ will be a solution to the original equation (**??**) only if it falls within the interval $\left(\phi_1^{1+\mu}, \phi_2^{-(1+\mu)}\right)$. The proof in Lemma 1 implies that in all cases when there is a unique solution $\tilde{z}$ to equation $\tilde{V}(z) = 0$, we necessarily have that $\tilde{V}(z) > 0$ for all $z < \tilde{z}$ and $\tilde{V}(z) < 0$ for all $z > \tilde{z}$. Therefore, in all cases with the unique solution $\tilde{z}$ to $V(z) = 0$, we have $\tilde{z} \in \left(\phi_1^{1+\mu}, \phi_2^{-(1+\mu)}\right)$ if and only if $\tilde{V}\left(\phi_1^{1+\mu}\right) > 0 > \tilde{V}\left(\phi_2^{-(1+\mu)}\right)$, which, after some algebra, gives condition (4).

Part (ii). Suppose that $\mu < 1$ and $0 < \overline{\beta} < 1$ and $0 < \gamma < 1$. From Lemma 1 we know that for any fixed $c = \phi_1/\phi_2$ there exists $\tilde{\tilde{\phi}}_2(c) > 0$ such that for all $\phi_2 < \tilde{\tilde{\phi}}_2(c)$ equation $\tilde{V}(z) = 0$ has three solutions. Denote these solutions as $\tilde{z}_1(\phi_2, c) < \tilde{z}_2(\phi_2, c) < \tilde{z}_3(\phi_2, c)$. Let us verify that for all low enough $\phi_2 > 0$ we have $\tilde{z}_1(\phi_2, c) < c^{1+\mu}\phi_2^{1+\mu} < \tilde{z}_2(\phi_2, c)$ and $\tilde{z}_2(\phi_2, c) < \phi_2^{-(1+\mu)} < \tilde{z}_3(\phi_2, c)$, which implies that $\tilde{z}_2(\phi_2, c)$ — and only $\tilde{z}_2(\phi_2, c)$ — results in a regular equilibrium of the economy.

The shape of $\tilde{V}(z)$ for $\phi_2 < \tilde{\tilde{\phi}}_2(c)$ implies that inequalities $\tilde{z}_1(\phi_2, c) < c^{1+\mu}\phi_2^{1+\mu} < \tilde{z}_2(\phi_2, c)$ hold if $\tilde{V}\left(c^{1+\mu}\phi_2^{1+\mu}\right) < 0$ and $c^{1+\mu}\phi_2^{1+\mu} < \overline{z}_0$, where $\overline{z}_0$ was defined in (8). These conditions, in turn, hold if

$$G > \max\left\{c^{1-\mu}\left[\zeta\gamma\overline{\beta}\phi_2^{\mu-1} + \left(1 - \zeta\gamma\overline{\beta}\right)c\phi_2^{1+\mu}\right]^{-1}, \frac{\overline{\gamma}}{\gamma}c^{1-\mu}\phi_2^{1-\mu}\right\}. \tag{10}$$

Similarly, the shape of $\tilde{V}(z)$ for $\phi_2 < \tilde{\tilde{\phi}}_2(c)$ implies that inequalities $\tilde{z}_2(\phi_2, c) < \phi_2^{-(1+\mu)} < \tilde{z}_3(\phi_2, c)$ hold if $\tilde{V}\left(\phi_2^{-(1+\mu)}\right) > 0$ and $\phi_2^{-(1+\mu)} > \overline{z}_0$. These conditions, in turn, hold if

$$G < \min\left\{\zeta\overline{\gamma}\overline{\beta}\phi_2^{\mu-1} + \left(1 - \zeta\overline{\gamma}\overline{\beta}\right)c\phi_2^{\mu+1}, \frac{\overline{\gamma}}{\gamma}\phi_2^{\mu-1}\right\}. \tag{11}$$

Observe that as $\phi_2 \to 0$, the right-hand side of inequality (10) goes to 0, while the right-hand side of inequality (11) goes to ∞. Thus, for all low enough ϕ_2 inequalities (10) and (11) hold. $\square$

To complete the proof of Proposition 1 we need to establish the stability results. As we show in Lemma 5 in Appendix B.6 of the main text, if $\tilde{z}$ solves equation $\tilde{V}(z) = 0$ and $\tilde{x}$ is a corresponding solution to $V(x) = 0$, then $dV(\tilde{x})/d\ln x < 0$ if and only if $dV(\tilde{z})/d\ln z < 0$.

The proofs of Lemmas 1 and 2 above imply that if $\phi_2 > \tilde{\tilde{\phi}}_2(c)$ then there exists a unique solution $\tilde{z} > 0$ to $\tilde{V}(z) = 0$, and this solution is such that $\tilde{V}'(\tilde{z}) < 0$. Thus, the corresponding equilibrium is unstable.

If $\phi_2 < \tilde{\tilde{\phi}}_2(c)$ then there are three solutions $0 < \tilde{z}_1 < \tilde{z}_2 < \tilde{z}_3$ to $\tilde{V}(z) = 0$. These solutions are such that $\tilde{V}'(\tilde{z}_1) < 0$, $\tilde{V}'(\tilde{z}_2) > 0$, and $\tilde{V}'(\tilde{z}_3) < 0$. Then, if $\phi_2 < \overline{\phi}_2(c)$ then only $\tilde{z}_2$ results in a solution to $V(x) = 0$, which corresponds to a locally stable equilibrium. If $\overline{\phi}_2(c) \leq \phi_2 < \tilde{\tilde{\phi}}_2(c)$ then equation $V(x) = 0$ can have at most three positive solutions corresponding to $\tilde{z}_1$, $\tilde{z}_2$, and $\tilde{z}_3$. Given the signs of the derivative $\tilde{V}'$ evaluated at $\tilde{z}_1$, $\tilde{z}_2$, and $\tilde{z}_3$, two of the corresponding regular equilibria are unstable and one is locally stable.

Finally, if $\phi_2 = \tilde{\tilde{\phi}}_2(c)$ and $c \neq \tilde{c}$ then there are two solutions $\tilde{z}_1 > 0$ and $\tilde{z}_2 > 0$ to $\tilde{V}(z) = 0$ with $\tilde{V}'(\tilde{z}_1) < 0$ and $\tilde{V}'(\tilde{z}_2) = 0$. In this case equation $V(x) = 0$ can have at most two solutions, one of which results in an unstable equilibrium and the other in an equilibrium that is neither stable nor unstable. If $\phi_2 = \tilde{\tilde{\phi}}_2(c)$ and $c = \tilde{c}$ then there is a unique solution $\tilde{z} > 0$ to $\tilde{V}(z) = 0$, which has $\tilde{V}'(\tilde{z}) = 0$. In this case equation $V(x) = 0$ can have at most one solution, which results in an equilibrium that is neither stable nor unstable.

This completes the proof of Proposition 1.

2 Exhaustive Analysis of Regular Equilibria in the Case with $\alpha = 1$

In the case with $\alpha = 1$ function $V(x)$ is given by

$$V(x) = \ln G - \ln \frac{\phi_1}{\phi_2} + (1-\alpha)\ln x + \mu \ln \frac{1+\phi_1 x}{\phi_2 + x} + \alpha \ln \frac{1+d_1 x}{d_2 + x}.$$

In what follows, it is a bit easier to work with the negative of $V(x)$, $\tilde{V}(x) \equiv -V(x)$. Taking the first derivative of $\tilde{V}$, we get

$$\tilde{V}'(x) = \frac{\mu(1-\phi_1\phi_2)}{(1+\phi_1 x)(\phi_2 + x)} + \frac{1-d_1 d_2}{(1+d_1 x)(d_2 + x)}.$$

We have $\tilde{V}'(x) = 0$ if and only if $W(x) = 0$, where $W(x) \equiv Ax^2 + Bx + C$ with

$$\begin{aligned} A &\equiv \mu(1-\phi_1\phi_2)d_1 + \phi_1(1-d_1d_2), \\ B &\equiv \mu(1-\phi_1\phi_2)(1+d_1d_2) + (1-d_1d_2)(1+\phi_1\phi_2), \\ C &\equiv \mu(1-\phi_1\phi_2)d_2 + (1-d_1d_2)\phi_2. \end{aligned}$$

Case 1. If $d_1d_2 \le 1$, then $W(x) > 0$ for all $x > 0$ and thus $\tilde{V}(x)$ is an increasing function. Hence, in this case there exists at most one regular equilibrium. This equilibrium exists if and only if $\tilde{V}(0) < 0 < \tilde{V}(\infty)$.

Case 2. Suppose that $d_1d_2 > 1$.

Case 2.1. Suppose that $A > 0$ and $C > 0$. In this case $W(x)$ is a convex function that achieves its minimum at $x^* = -\frac{B}{2A}$.

Case 2.1.1. If $B^2 \le 4AC$ then $W(x) > 0$ for all x except for, maybe, $x = -B/(2A)$. In this case $\tilde{V}(x)$ is increasing. As in Case 1, there exists a unique regular equilibrium if $\tilde{V}(0) < 0 < \tilde{V}(\infty)$, otherwise there are no regular equilibria.

Case 2.1.2. If $B > 2\sqrt{AC}$, then $W(x) < 0$ if and only if $x \in (x_1^*, x_2^*)$, where

$$x_1^* = \frac{-B - \sqrt{B^2 - 4AC}}{2A} \quad \text{and} \quad x_2^* = \frac{-B + \sqrt{B^2 - 4AC}}{2A}.$$

Having $B > 2\sqrt{AC}$ implies that $x_1^* < x_2^* < 0$ and that $W(x) > 0$ for $x > 0$. Hence, again, there exists a unique regular equilibrium if $\tilde{V}(0) < 0 < \tilde{V}(\infty)$, otherwise there are no regular equilibria.

Case 2.1.3. Suppose that $B < -2\sqrt{AC}$. Then $0 < x_1^* < x_2^*$ and, therefore, $\tilde{V}(x)$ is increasing for $x < x_1^*$ and $x > x_2^*$, and $\tilde{V}(x)$ is decreasing for $x \in (x_1^*, x_2^*)$.

Case 2.1.3.1. Suppose that $\tilde{V}(0) < 0 < \tilde{V}(\infty)$. If $\tilde{V}(x_1^*) < 0$ then there is a unique equilibrium. If $\tilde{V}(x_1^*) = 0$, then there are two equilibria. And if $\tilde{V}(x_1^*) > 0$, then there are three equilibria.

Case 2.1.3.2. Suppose that $\tilde{V}(\infty) \le 0 \le \tilde{V}(0)$. Then there is a unique equilibrium.

Case 2.1.3.3. Suppose that $\tilde{V}(0) < 0$ and $\tilde{V}(\infty) \le 0$. If $\tilde{V}(x_1^*) < 0$ then there are no regular equilibria. If $\tilde{V}(x_1^*) = 0$, then there is a unique regular equilibrium. If $\tilde{V}(x_1^*) > 0$ then there are two regular equilibria.

Case 2.1.3.4. Suppose that $\tilde{V}(0) \ge 0$ and $\tilde{V}(\infty) > 0$. If $\tilde{V}(x_2^*) > 0$ then there are no regular equilibria. If $\tilde{V}(x_2^*) = 0$, then there is a unique regular equilibrium. If $\tilde{V}(x_2^*) < 0$ then there are two regular equilibria.

Case 2.2. Suppose that $A > 0$ and $C \le 0$. In this case $x_1^* \le 0 \le x_2^*$. Then $W(x) < 0$ for

$0 < x < x_2^*$ and $W(x) > 0$ for $x > x_2^*$. Thus, x_2^* is the global minimum of $\tilde{V}(x)$.

Case 2.2.1. If $\tilde{V}(0) \leq 0 < \tilde{V}(\infty)$, then there is a unique equilibrium.

Case 2.2.2. If $\tilde{V}(\infty) \leq 0 < \tilde{V}(0)$, then there is a unique equilibrium.

Case 2.2.3. If $\tilde{V}(0) \leq 0$ and $\tilde{V}(\infty) \leq 0$, then there are no regular equilibria.

Case 2.2.4. Suppose that $\tilde{V}(0) > 0$ and $\tilde{V}(\infty) > 0$. If $\tilde{V}(x_2^*) > 0$ then there are no regular equilibria. If $\tilde{V}(x_2^*) = 0$, then there is a unique regular equilibrium. If $\tilde{V}(x_2^*) < 0$ then there are two regular equilibria.

Case 2.3: Suppose that $A < 0$ and $C < 0$. In this case $W(x)$ is a concave function that achieves its maximum at $x^* = -\frac{B}{2A}$.

Case 2.3.1. If $B^2 \leq 4AC$ then $W(x) < 0$ for all x except for, maybe, $x = -B/(2A)$. In this case $\tilde{V}(x)$ is decreasing. There exists a unique regular equilibrium if $\tilde{V}(\infty) < 0 < \tilde{V}(0)$, otherwise there are no regular equilibria.

Case 2.3.2. If $B > 2\sqrt{AC}$, then $x_1^* < x_2^* < 0$ and $W(x) < 0$ for $x > 0$. Hence, again, there exists a unique regular equilibrium if $\tilde{V}(\infty) < 0 < \tilde{V}(0)$, otherwise there are no regular equilibria.

Case 2.3.3. Suppose that $B < -2\sqrt{AC}$. Then $0 < x_1^* < x_2^*$, and $\tilde{V}(x)$ is decreasing for $x < x_1^*$ and $x > x_2^*$, and $\tilde{V}(x)$ is increasing for $x \in (x_1^*, x_2^*)$.

Case 2.3.3.1. Suppose that $\tilde{V}(\infty) < 0 < \tilde{V}(0)$. Then if $\tilde{V}(x_1^*) > 0$ then there is a unique equilibrium. If $\tilde{V}(x_1^*) = 0$, then there are two equilibria. And if $\tilde{V}(x_1^*) < 0$, then there are three equilibria.

Case 2.3.3.2. If $\tilde{V}(0) \leq 0 \leq \tilde{V}(\infty)$ then there is a unique equilibrium.

Case 2.3.3.3. Suppose that $\tilde{V}(0) > 0$ and $\tilde{V}(\infty) \geq 0$. If $\tilde{V}(x_1^*) > 0$ then there are no regular equilibria. If $\tilde{V}(x_1^*) = 0$, then there is a unique regular equilibrium. If $\tilde{V}(x_1^*) < 0$ then there are two regular equilibria.

Case 2.3.3.4. Suppose that $\tilde{V}(0) \leq 0$ and $\tilde{V}(\infty) < 0$. If $\tilde{V}(x_2^*) < 0$ then there are no regular equilibria. If $\tilde{V}(x_2^*) = 0$, then there is a unique regular equilibrium. If $\tilde{V}(x_2^*) > 0$ then there are two regular equilibria.

Case 2.4. Suppose that $A < 0$ and $C \geq 0$. In this case $x_1^* \leq 0 \leq x_2^*$. Then $W(x) > 0$ for $0 < x < x_2^*$ and $W(x) < 0$ for $x > x_2^*$. Thus, x_2^* is the global maximum of $\tilde{V}(x)$.

Case 2.4.1. If $\tilde{V}(\infty) < 0 \leq \tilde{V}(0)$ then there is a unique equilibrium.

Case 2.4.2. If $\tilde{V}(0) < 0 \leq \tilde{V}(\infty)$ then there is a unique equilibrium.

Case 2.4.3. If $\tilde{V}(0) \geq 0$ and $\tilde{V}(\infty) \geq 0$ then there are no regular equilibria.

Case 2.4.4. Suppose that $\tilde{V}(0) < 0$ and $\tilde{V}(\infty) < 0$. If $\tilde{V}(x_2^*) < 0$ then there are no regular equilibria. If $\tilde{V}(x_2^*) = 0$, then there is a unique regular equilibrium. If $\tilde{V}(x_2^*) > 0$ then there are two regular equilibria.

Case 2.5. Suppose that $A = 0$. In this case, $W(x) = Bx + C$.

Case 2.5.1. If $B < 0$ and $C \geq 0$ then $\tilde{V}(x)$ is increasing for $x < x^*$ and decreasing for $x > x^* \equiv -C/B$. This is the same as case 2.4.

Case 2.5.2. If $B > 0$ and $C \leq 0$ then $\tilde{V}(x)$ is decreasing for $x < x^*$ and increasing for $x > x^* \equiv -C/B$. This is the same as case 2.2.

Case 2.5.3. Suppose that $B < 0$ and $C \leq 0$. Then $\tilde{V}(x)$ is decreasing for all $x > 0$. If $\tilde{V}(\infty) < 0 < \tilde{V}(0)$, then there exists a unique equilibrium, otherwise there are no regular equilibria.

Case 2.5.4. Suppose that $B > 0$ and $C \geq 0$. Then $\tilde{V}(x)$ is increasing for all $x > 0$. If $\tilde{V}(0) < 0 < \tilde{V}(\infty)$ then there exists a unique equilibrium, otherwise there are no regular equilibria.

Now we can collect outcomes of all cases above and write conditions for different number of regular equilibria. Let us denote:

$$F(x) \equiv G^{-1}\left(\frac{\phi_1}{\phi_2}\right)\left(\frac{1+\phi_1 x}{\phi_2 + x}\right)^{\mu}\left(\frac{1+d_1 x}{d_2 + x}\right)^{-1};$$

$$\begin{aligned} F_0 &\equiv \lim_{x\to 0} F(x) = G^{-1}\left(\frac{\phi_1}{\phi_2}\right)\phi_2^{\mu} d_2, \\ F_\infty &\equiv \lim_{x\to\infty} F(x) = G^{-1}\left(\frac{\phi_1}{\phi_2}\right)\phi_1^{-\mu} d_1^{-1}; \end{aligned}$$

$$\begin{aligned} \bar{x}_1 &\equiv \frac{-B - \sqrt{B^2 - 4AC}}{2A} \text{ for } A \neq 0; \\ \bar{x}_2 &\equiv \begin{cases} \dfrac{-B + \sqrt{B^2 - 4AC}}{2A} & \text{if } A > 0, \\ -\dfrac{C}{B} & \text{if } A = 0. \end{cases} \end{aligned}$$

1. **No regular equilibria:**
   (a) $d_1 d_2 \leq 1$ and either $F_0 \geq 1$ or $F_\infty \leq 1$;
   (b) $d_1 d_2 > 1$, $A > 0$, $C > 0$, and one of the following conditions:
      i. $B \geq -2\sqrt{AC}$, and either $F_0 \geq 1$ or $F_\infty \leq 1$; or
      ii. $B < -2\sqrt{AC}$, $F_0 < 1$ and $F_\infty \leq 1$, and $F(\bar{x}_1) < 1$; or
      iii. $B < -2\sqrt{AC}$, $F_0 \geq 1$ and $F_\infty > 1$, and $F(\bar{x}_2) > 1$;
   (c) $d_1 d_2 > 1$, $A < 0$, $C < 0$, and one of the following conditions:

      i. $B \geq -2\sqrt{AC}$, and either $F_0 \leq 1$ or $F_\infty \geq 1$; or
      ii. $B < -2\sqrt{AC}$, $F_0 > 1$ and $F_\infty \geq 1$, and $F(\bar{x}_1) > 1$; or
      iii. $B < -2\sqrt{AC}$, $F_0 \leq 1$ and $F_\infty < 1$, and $F(\bar{x}_2) < 1$;
   (d) $d_1 d_2 > 1$, $A > 0$, $C \leq 0$, and one of the following conditions:
      i. $F_0 \leq 1$ and $F_\infty \leq 1$; or
      ii. $F_0 > 1$ and $F_\infty > 1$, and $F(\bar{x}_2) > 1$;
   (e) $d_1 d_2 > 1$, $A = 0$, $B > 0$, $C \leq 0$, and one of the conditions (1.d.i)-(1.d.ii).
   (f) $d_1 d_2 > 1$, $A < 0$, $C \geq 0$, and one of the following conditions:
      i. $F_0 \geq 1$ and $F_\infty \geq 1$; or
      ii. $F_0 < 1$ and $F_\infty < 1$, and $F(\bar{x}_2) < 1$;
   (g) $d_1 d_2 > 1$, $A = 0$, $B < 0$, $C \geq 0$, and one of the conditions (1.f.i)-(1.f.ii).

2. **Unique regular equilibrium:**
   (a) $d_1 d_2 \leq 1$ and $F_0 < 1 < F_\infty$;
   (b) $d_1 d_2 > 1$, $A > 0$, $C > 0$, and one the following conditions:
      i. $B \geq -2\sqrt{AC}$ and $F_0 < 1 < F_\infty$; or
      ii. $B < -2\sqrt{AC}$, $F_0 < 1 < F_\infty$, and $F(\bar{x}_1) < 1$; or
      iii. $B < -2\sqrt{AC}$ and $F_\infty \leq 1 \leq F_0$; or
      iv. $B < -2\sqrt{AC}$, $F_0 < 1$ and $F_\infty \leq 1$, and $F(\bar{x}_1) = 1$; or
      v. $B < -2\sqrt{AC}$, $F_0 \geq 1$ and $F_\infty > 1$, and $F(\bar{x}_2) = 1$;
   (c) $d_1 d_2 > 1$, $A < 0$, $C < 0$, and one the following conditions:
      i. $B \geq -2\sqrt{AC}$ and $F_\infty < 1 < F_0$; or
      ii. $B < -2\sqrt{AC}$, $F_\infty < 1 < F_0$, and $F(\bar{x}_1) > 1$; or
      iii. $B < -2\sqrt{AC}$, $F_0 \leq 1 \leq F_\infty$; or
      iv. $B < -2\sqrt{AC}$, $F_0 > 1$ and $F_\infty \geq 1$, and $F(\bar{x}_1) = 1$; or
      v. $B < -2\sqrt{AC}$, $F_0 \leq 1$ and $F_\infty < 1$, and $F(\bar{x}_2) = 1$;
   (d) $d_1 d_2 > 1$, $A > 0$, $C \leq 0$, and one the following conditions:
      i. $F_0 \leq 1 < F_\infty$; or
      ii. $F_\infty \leq 1 < F_0$; or
      iii. $F_0 > 1$ and $F_\infty > 1$, and $F(\bar{x}_2) = 1$;
   (e) $d_1 d_2 > 1$, $A = 0$, $B > 0$, $C \leq 0$, and one of the conditions (2.d.i)-(2.d.iii).

  - (f) $d_1 d_2 > 1$, $A < 0$, $C \geq 0$, and one the following conditions:
    - i. $F_\infty < 1 \leq F_0$; or
    - ii. $F_0 < 1 \leq F_\infty$; or
    - iii. $F_0 < 1$ and $F_\infty < 1$, and $F(\bar{x}_2) = 1$;
  - (g) $d_1 d_2 > 1$, $A = 0$, $B < 0$, $C \geq 0$, and one of the conditions (2.f.i)-(2.f.iii).

3. **Two regular equilibria:**
  - (a) $d_1 d_2 > 1$, $A > 0$, $C > 0$, and one of the following conditions:
    - i. $B < -2\sqrt{AC}$, $F_0 < 1 < F_\infty$, and $F(\bar{x}_1) = 1$; or
    - ii. $B < -2\sqrt{AC}$, $F_0 < 1$ and $F_\infty \leq 1$, and $F(\bar{x}_1) > 1$; or
    - iii. $B < -2\sqrt{AC}$, $F_0 \geq 1$ and $F_\infty > 1$, and $F(\bar{x}_2) < 1$;
  - (b) $d_1 d_2 > 1$, $A < 0$, $C < 0$, and one of the following conditions:
    - i. $B < -2\sqrt{AC}$, $F_\infty < 1 < F_0$ and $F_\infty < 1$, and $F(\bar{x}_1) = 1$; or
    - ii. $B < -2\sqrt{AC}$, $F_0 > 1$ and $F_\infty \geq 1$, and $F(\bar{x}_1) < 1$; or
    - iii. $B < -2\sqrt{AC}$, $F_0 \leq 1$ and $F_\infty < 1$, and $F(\bar{x}_2) > 1$;
  - (c) $d_1 d_2 > 1$, $A > 0$, $C \leq 0$, and $F_0 > 1$ and $F_\infty > 1$, and $F(\bar{x}_2) < 1$;
  - (d) $d_1 d_2 > 1$, $A = 0$, $B > 0$, $C \leq 0$, and $F_0 > 1$ and $F_\infty > 1$, and $F(\bar{x}_2) < 1$;
  - (e) $d_1 d_2 > 1$, $A < 0$, $C \geq 0$, and $F_0 < 1$ and $F_\infty < 1$, and $F(\bar{x}_2) > 1$;
  - (f) $d_1 d_2 > 1$, $A = 0$, $B < 0$, $C \geq 0$, and $F_0 < 1$ and $F_\infty < 1$, and $F(\bar{x}_2) > 1$;

4. **Three regular equilibria:**
  - (a) $d_1 d_2 > 1$, $A > 0$, $C > 0$, $B < -2\sqrt{AC}$, $F_0 < 1 < F_\infty$, and $F(\bar{x}_1) > 1$;
  - (b) $d_1 d_2 > 1$, $A < 0$, $C < 0$, $B < -2\sqrt{AC}$, $F_\infty < 1 < F_0$, and $F(\bar{x}_1) < 1$.

3 Uniqueness Boundary for the AA Economy with $0 < \alpha < 1$

The following proposition gives an alternative perspective on conditions in Proposition 5 in the main text.

Proposition 2 (AA economy: Uniqueness boundary for $0 < \alpha < 1$). *Assume that* $0 < \alpha < 1$, $\beta = 1$, *and* $\phi_1, \phi_2 \in (0, \infty)$. *Then for any fixed* $c \equiv \phi_1 / \phi_2$*:*

(a) *If* $\phi_2 \geq \tilde{\phi}_2(c)$*, where*

$$\tilde{\phi}_2(c) \equiv \eta \cdot \min\left\{\left[cG^{-1}\right]^{-\frac{1}{\mu+1}}, \left[c^{\mu}G\right]^{-\frac{1}{\mu+1}}\right\}, \tag{12}$$

and $\eta \equiv \frac{\mu+2\alpha-1}{\mu+1}$*, then the equilibrium is unique. Moreover, in the symmetric case with* $c = 1$ *and* $G = 1$ *condition* $\phi_2 \geq \tilde{\phi}_2(c)$ *is both necessary and sufficient for uniqueness of equilibrium.*

(b) *If* $\phi_2 < \tilde{\phi}_2(c)$ *then there exists* $\tilde{\tilde{\phi}}_2(c) \in (0, \tilde{\phi}_2(c)]$ *such that the equilibrium is unique if* $\phi_2 > \tilde{\tilde{\phi}}_2(c)$*, whereas there are three equilibria if* $\phi_2 < \tilde{\tilde{\phi}}_2(c)$*. If* $\phi_2 = \tilde{\tilde{\phi}}_2(c)$*, then generically there are two equilibria except for the special case when* $G = c^{\frac{1-\mu}{2}}$*, in which case there is a unique equilibrium. Moreover, if* $G = c^{\frac{1-\mu}{2}}$ *then* $\tilde{\tilde{\phi}}_2(c) = \tilde{\phi}_2(c)$*.*

Part (b) of Proposition 2 immediately implies that in the symmetric case ($\phi_1 = \phi_2$ and $G = 1$) condition $\phi_2 \geq \eta \equiv \frac{\mu+2\alpha-1}{\mu+1}$ is both necessary and sufficient for uniqueness of equilibrium. We state this fact in the following corollary.

Corollary 1 (**AA economy: $0 < \alpha < 1$, symmetric case**). *Assume that* $0 < \alpha < 1$, $\beta = 1$, $\phi_1 = \phi_2 \leq 1$*, and* $G = 1$*. Then the economy has a unique equilibrium if and only if* $\phi_2 \geq \eta \equiv \frac{\mu+2\alpha-1}{\mu+1}$.

It is easy to check that in the symmetric case with $\beta = 1$ condition $\phi_2 \geq \eta$ is equivalent to condition $\mu \leq \overline{\mu}_a$, where $\overline{\mu}_a \equiv \dfrac{1+\sqrt{\phi_1\phi_2}}{1-\sqrt{\phi_1\phi_2}} \cdot \dfrac{1-2\alpha+\sqrt{d_1d_2}}{1+\sqrt{d_1d_2}}$ was defined in Appendix A of the main text.[1] In its turn, condition $\mu \leq \overline{\mu}_a$ is the (sufficient) condition from part (ii.a) of Proposition 3 and part (a.1) of Proposition 5 from the main text.

Observe that in Proposition 2 we drop the assumption $\phi_1 \leq 1$ and $\phi_2 \leq 1$ and allow for any positive ϕ_1 and ϕ_2, which means that we allow the iceberg trade costs to be negative. This simplifies the formulation and proof of Proposition 2, while accommodating the restriction $\phi_1 \leq 1$ and $\phi_2 \leq 1$ is straightforward: we just need to intersect the region given by this restriction with the regions with unique equilibrium and three equilibria outlined by the uniqueness boundary.

Proposition 2 implies that we can trace the "uniqueness boundary" in the space $\phi_1 > 0$ and $\phi_2 > 0$ by varying the ratio $c = \phi_1/\phi_2$ between 0 and $+\infty$. A typical uniqueness boundary for the case with $G = 1$ is illustrated in Figure 4. In this figure, we depict two threshold values for ϕ_2, $\tilde{\tilde{\phi}}_2(c')$ and $\tilde{\tilde{\phi}}_2(c'')$, that correspond to two fixed levels of the ratio ϕ_1/ϕ_2, $c' = 1$ and $c'' < 1$. Proposition 2 says that any ray corresponding to a fixed ratio ϕ_1/ϕ_2 intersects the uniqueness boundary only once. At the same time, as the right part of Figure 4 shows, lines with fixed levels of one of the trade costs can intersect

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[1] Recall that with $\beta = 1$ we have $d_i = \phi_i$ for $i = 1, 2$.

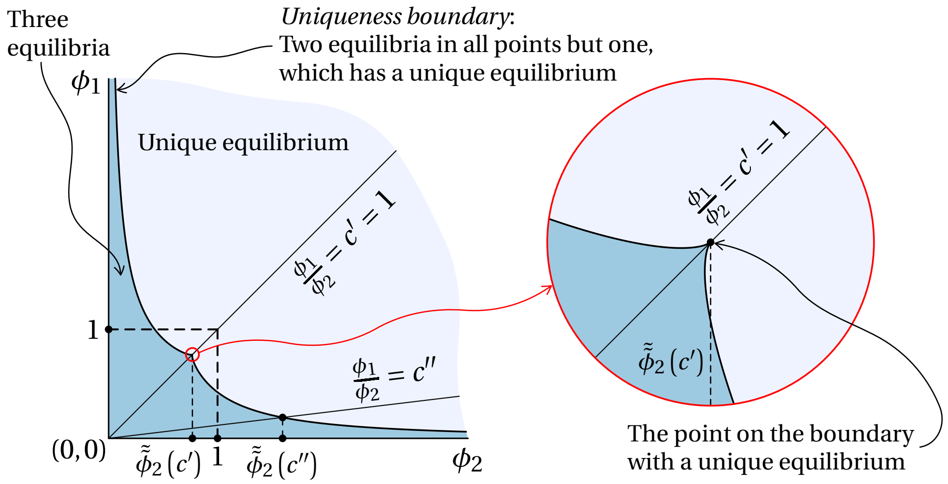

Figure 4: Illustration to Proposition 2 for AA economy: uniqueness boundary.

the uniqueness boundary several times. In particular, the vertical line at $\phi_2 = \tilde{\tilde{\phi}}_2(c')$ in Figure 4 intersects the uniqueness boundary two times.

According to Proposition 2, we generically have two equilibria — an even number — on the uniqueness boundary. At the same time, the Index Theorem (Kehoe, 1980) implies that an economy generically has an odd number of equilibria. The reason for having an even number of equilibria on the uniqueness boundary is that the Jacobian of the equilibrium system is singular in one of the equilibrium points with parameters on the uniqueness boundary. Thus, conditions of the Index Theorem are not satisfied in this case. Intuitively, as we change trade costs to go from the uniqueness area to the multiplicity area, we get a new equilibrium point on the uniqueness boundary. This new equilibrium point is associated with a singular Jacobian of the equilibrium system of equations. After we cross the uniqueness boundary, this new equilibrium point is split into two different equilibrium points each of which is associated with non-singular Jacobians of the equilibrium system. In the special case with $G = c^{\frac{1-\mu}{2}}$, the two equilibrium points on the uniqueness boundary happen to coincide, which gives us a unique equilibrium. In this case, as we change trade costs to go to the multiplicity area, the point on the uniqueness boundary is split into three equilibrium points.

Let us now turn to the proof of Proposition 2. In the case with $\beta = 1$, expressions for d_1 and d_2 collapse to $d_1 = \phi_1$ and $d_2 = \phi_2$, and function $V(x)$ becomes

$$V(x) = \ln G - \alpha \ln \frac{\phi_1}{\phi_2} + (1-\alpha)\ln x + (\alpha+\mu)\ln\left(\frac{1+\phi_1 x}{\phi_2 + x}\right).$$

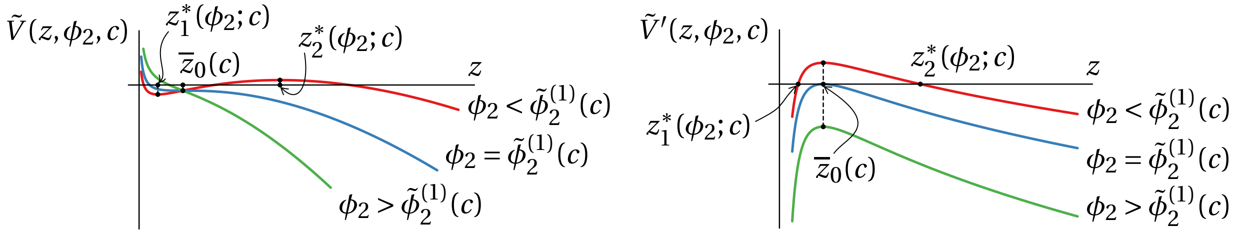

Figure 5: AA economy, $0 < \alpha < 1$: $\tilde{V}$ and $\tilde{V}'$

Introducing the change of variables $z = x^{1-\frac{1-\alpha}{\alpha+\mu}}$, using $\phi_1 = c\phi_2$, and after doing some algebra, we get equation $\tilde{V}\left(z, \phi_2, c\right) = 0$, where

$$\begin{aligned}\tilde{V}\left(z, \phi_2, c\right) &\equiv \left(\phi_2 z^{1-\frac{\alpha+\mu}{2\alpha+\mu-1}} + z\right) \cdot \left(1 - \exp\left\{\frac{1}{\alpha+\mu} V\left(z^{\frac{\alpha+\mu}{2\alpha+\mu-1}}\right)\right\}\right) \\ &= \phi_2 z^{1-\frac{\alpha+\mu}{2\alpha+\mu-1}} - \phi_2 c^{\frac{\mu}{\alpha+\mu}} \tilde{G} z^{\frac{\alpha+\mu}{2\alpha+\mu-1}} + z - \tilde{G} c^{-\frac{\alpha}{\alpha+\mu}}\end{aligned}$$

and $\tilde{G} \equiv G^{\frac{1}{\alpha+\mu}}$. Observe that $1 - \frac{1-\alpha}{\alpha+\mu} = \frac{2\alpha+\mu-1}{\alpha+\mu} = \left(2 + 1/\varepsilon\right) \frac{\alpha}{\alpha+\mu} > 0$ and so the change of variables $z = x^{1-\frac{\mu-\alpha\mu}{\alpha\mu+1}}$ is well-defined. The first and the second derivatives of $\tilde{V}\left(z, \phi_2, c\right)$ with respect to z are given by

$$\begin{aligned}\tilde{V}_1'\left(z, \phi_2, c\right) &= -\frac{1-\alpha}{2\alpha+\mu-1} \phi_2 z^{-\frac{\alpha+\mu}{2\alpha+\mu-1}} - \frac{\alpha+\mu}{2\alpha+\mu-1} \phi_2 c^{\frac{\mu}{\mu+\alpha}} \tilde{G} z^{\frac{1-\alpha}{2\alpha+\mu-1}} + 1, \\ \tilde{V}_1''\left(z, \phi_2, c\right) &= \frac{\left(1-\alpha\right)\left(\alpha+\mu\right)}{\left(2\alpha+\mu-1\right)^2} \phi_2 z^{\frac{1-\alpha}{2\alpha+\mu-1}-1} \left(z^{-\frac{\mu+1}{2\alpha+\mu-1}} - \left[\overline{z}_0\left(c\right)\right]^{-\frac{\mu+1}{2\alpha+\mu-1}}\right),\end{aligned}$$

where

$$\overline{z}_0\left(c\right) \equiv \left(c^{\frac{\mu}{\alpha+\mu}} \tilde{G}\right)^{-\frac{2\alpha+\mu-1}{\mu+1}}. \tag{13}$$

From here we see that $\tilde{V}_1''\left(z, \phi_2, c\right) > 0$ if and only if $z < \overline{z}_0\left(c\right)$. Thus, $\tilde{V}_1'\left(z, \phi_2, c\right)$ is a concave function in z that achieves its maximum at $\overline{z}_0\left(c\right)$. Evaluating $\tilde{V}_1'\left(\overline{z}_0\left(c\right), \phi_2, c\right)$, we get

$$\tilde{V}_1'\left(\overline{z}_0\left(c\right), \phi_2, c\right) = -\frac{\mu+1}{2\alpha+\mu-1} \phi_2 c^{\frac{\mu}{\mu+1}} \tilde{G}^{\frac{\alpha+\mu}{\mu+1}} + 1,$$

and so $\tilde{V}_1'\left(\overline{z}_0\left(c\right), \phi_2, c\right) < 0$ if and only if $\phi_2 > \tilde{\phi}_2^{(1)}\left(c\right)$, where

$$\tilde{\phi}_2^{(1)}\left(c\right) \equiv \frac{2\alpha+\mu-1}{\mu+1} c^{-\frac{\mu}{\mu+1}} \tilde{G}^{-\frac{\alpha+\mu}{\mu+1}}.$$

If $\phi_2 > \tilde{\phi}_2^{(1)}\left(c\right)$, then $\tilde{V}_1'\left(\overline{z}_0\left(c\right), \phi_2, c\right) < 0$ and, hence, $\tilde{V}_1'\left(z, \phi_2, c\right) < 0$ for all z, and so function $\tilde{V}\left(z, \phi_2, c\right)$ is decreasing in z (see Figure 5 for illustration). Then, given that $\lim_{z\to 0}\tilde{V}(z,\phi_2,c) = \infty$ and $\lim_{z\to\infty}\tilde{V}(z,\phi_2,c) = -\infty$, we conclude that function $\tilde{V}(z,\phi_2,c)$ intersects the horizontal axis $z = 0$ once and only once for some $\tilde{z} > 0$. The case with $\phi_2 = \tilde{\phi}_2^{(1)}(c)$ is similar to the case with $\phi_2 > \tilde{\phi}_2^{(1)}(c)$ with the only difference that function $\tilde{V}\left(z,\tilde{\phi}_2^{(1)}(c),c\right)$ is decreasing for all z except for $z = \bar{z}_0(c)$. In this case as well $\tilde{V}(z,\phi_2,c)$ intersects the horizontal axis $z = 0$ only once for some $\tilde{z} > 0$.

Let us now write $\tilde{V}(z,\phi_2,c)$ as $\tilde{V}(z,\phi_2,c) = -c^{-\frac{\alpha}{\mu+\alpha}}\tilde{G}z\tilde{\tilde{V}}\left(z^{-1},\phi_1,c\right)$, where

$$\tilde{\tilde{V}}(z,\phi_1,c) \equiv \phi_1 z^{1-\frac{\alpha+\mu}{2\alpha+\mu-1}} - \phi_1 c^{-\frac{\mu}{\mu+\alpha}}\tilde{G}^{-1}z^{\frac{\alpha+\mu}{2\alpha+\mu-1}} + z - \tilde{G}^{-1}c^{\frac{\alpha}{\mu+\alpha}}.$$

Clearly, functions $\tilde{V}(z,\phi_2,c)$ and $\tilde{\tilde{V}}(z,\phi_1,c)$ have the same number of zeros for $z > 0$. Moreover, function $\tilde{\tilde{V}}(z,\phi_1,c)$ is similar to function $\tilde{V}(z,\phi_2,c)$ with the difference that ϕ_1 is swapped with ϕ_2, c^{-1} is swapped with c, and $\tilde{G}^{-1}$ is swapped with $\tilde{G}$. Repeating for $\tilde{\tilde{V}}(z,\phi_1,c)$ the same analysis as for $\tilde{V}(z,\phi_2,c)$ above, we get that if $\phi_1 \geq \tilde{\phi}_1^{(2)}(c)$, where

$$\tilde{\phi}_1^{(2)}(c) \equiv \frac{2\alpha+\mu-1}{\mu+1}c^{\frac{\mu}{\mu+1}}\tilde{G}^{\frac{\alpha+\mu}{\mu+1}},$$

then $\tilde{\tilde{V}}(z,\phi_1,c)$ has a unique solution. Having $\phi_1 \geq \tilde{\phi}_1^{(2)}(c)$ for a particular c is equivalent to having $\phi_2 \geq \tilde{\phi}_2^{(2)}(c)$ for this c, where

$$\tilde{\phi}_2^{(2)}(c) \equiv c^{-1}\tilde{\phi}_1^{(2)}(c) = \frac{2\alpha+\mu-1}{\mu+1}c^{-\frac{1}{\mu+1}}\tilde{G}^{\frac{\alpha+\mu}{\mu+1}}.$$

Thus, we get that $\tilde{V}(z,\phi_2,c) = 0$ has a unique solution if $\phi_2 \geq \tilde{\phi}_2^{(1)}(c)$ or $\phi_2 \geq \tilde{\phi}_2^{(2)}(c)$. This is equivalent to

$$\phi_2 \geq \tilde{\phi}_2(c) \equiv \min\left\{\tilde{\phi}_2^{(1)}(c), \tilde{\phi}_2^{(2)}(c)\right\} = \frac{2\alpha+\mu-1}{\mu+1}\min\left\{\left[c^{\mu}G\right]^{-\frac{1}{\mu+1}}, \left[cG^{-1}\right]^{-\frac{1}{\mu+1}}\right\},$$

where we used the definition of $\tilde{G} = G^{\frac{1}{\alpha+\mu}}$. This proves part (a) of Proposition 2.

For later use, note that $\tilde{\tilde{V}}(z,\phi_1,c)$ is a decreasing function of z if $\phi_1 > \tilde{\phi}_1^{(2)}(c)$ or, equivalently, if $\phi_2 > \tilde{\phi}_2^{(2)}(c)$. This means that $\tilde{\tilde{V}}\left(z^{-1},\phi_1,c\right)$ is an increasing function of z if $\phi_2 > \tilde{\phi}_2^{(2)}(c)$ and, since $\tilde{V}(z,\phi_2,c) = -c^{-\frac{\alpha}{\mu+\alpha}}\tilde{G}z\tilde{\tilde{V}}\left(z^{-1},\phi_1,c\right)$, we have that $\tilde{V}(z,\phi_2,c)$ is a decreasing function of z if $\phi_2 > \tilde{\phi}_2^{(2)}(c)$. Earlier we argued that $\tilde{V}(z,\phi_2,c)$ is a decreasing function of z if $\phi_2 > \tilde{\phi}_2^{(1)}(c)$. This means that $\tilde{V}(z,\phi_2,c)$ is a decreasing function of z if $\phi_2 > \tilde{\phi}_2(c)$.

For the rest of this proof, we focus on the case with $\phi_2 < \tilde{\phi}_2(c)$ and prove part (b) of Proposition 2. We divide this proof into two steps.

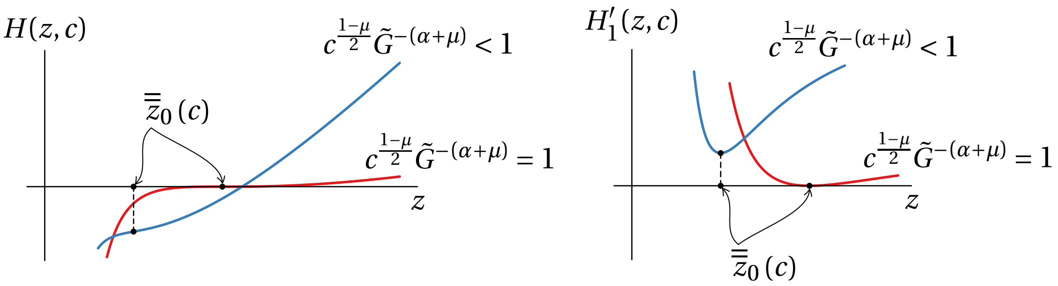

Figure 6: No agricultural sector, $0 < \alpha < 1$: H and H'

Step 1. We first prove that the system of equations $\tilde{V}(z, \phi_2, c) = 0$ and $\tilde{V}_1'(z, \phi_2, c) = 0$ in z and ϕ_2 has a unique solution $\tilde{\tilde{z}}(c) > 0$ and $0 < \tilde{\tilde{\phi}}_2(c) \leq \tilde{\phi}_2(c)$.

Solving for ϕ_2 from equation $\tilde{V}_1'(z, \phi_2, c) = 0$, substituting the result into equation $\tilde{V}(z, \phi_2, c) = 0$, and after doing some algebra, we get equation $H(z,c) = 0$, where

$$H(z,c) \equiv \tilde{G}^{-1} z^{-\frac{2(1-\alpha)}{2\alpha+\mu-1}} - \frac{1-\alpha}{\alpha+\mu} c^{-\frac{\alpha}{\alpha+\mu}} z^{-\frac{\mu+1}{2\alpha+\mu-1}} + \frac{1-\alpha}{\alpha+\mu} c^{\frac{\mu}{\alpha+\mu}} z - c^{\frac{\mu-\alpha}{\alpha+\mu}} \tilde{G}.$$

We are going to show that for any $c > 0$ there is a unique $\tilde{\tilde{z}}(c) > 0$ such that $H(\tilde{\tilde{z}}(c), c) = 0$.

The first and the second derivatives of $H(z,c)$ with respect to z are given by

$$\begin{aligned}
H_1'(z,c) &= \frac{1-\alpha}{2\alpha+\mu-1}\left(-2\tilde{G}^{-1} z^{-\frac{\mu+1}{2\alpha+\mu-1}} + \frac{\mu+1}{\alpha+\mu} c^{-\frac{\alpha}{\alpha+\mu}} z^{-\frac{2(\alpha+\mu)}{2\alpha+\mu-1}}\right) + \frac{1-\alpha}{\alpha+\mu} c^{\frac{\mu}{\alpha+\mu}}, \\
H_1''(z,c) &= \frac{2(1-\alpha)(\mu+1)}{(2\alpha+\mu-1)^2} \tilde{G}^{-1} z^{-\frac{2(\alpha+\mu)}{2\alpha+\mu-1}-1} \left(z - \bar{\bar{z}}_0(c)\right).
\end{aligned}$$

where $\bar{\bar{z}}_0(c) \equiv \tilde{G} c^{-\frac{\alpha}{\alpha+\mu}}$. From here we see that $H_1''(z,c) > 0$ if and only if $z > \bar{\bar{z}}_0(c)$. Thus, for any $c > 0$, $H_1'(z,c)$ is a convex function in z, and $\bar{\bar{z}}_0(c)$ is its minimum (see Figure 6 for illustration).

Evaluating $H_1'(\bar{\bar{z}}_0(c), c)$, we get

$$H_1'(\bar{\bar{z}}_0(c), c) = -\frac{1-\alpha}{\alpha+\mu} c^{\frac{\mu}{\alpha+\mu}} \left(\left(c^{\frac{1-\mu}{2}} \tilde{G}^{-(\alpha+\mu)}\right)^{\frac{2}{2\alpha+\mu-1}} - 1\right),$$

and so $H_1'(\bar{\bar{z}}_0(c), c) > 0$ if and only if $c^{\frac{1-\mu}{2}} \tilde{G}^{-(\alpha+\mu)} < 1$. From here we immediately see that if $c^{\frac{1-\mu}{2}} \tilde{G}^{-(\alpha+\mu)} < 1$, then $H_1'(z,c) > 0$ for all z, and so $H(z,c)$ increases in z. Then, given that $\lim_{z\to 0} H(z,c) = -\infty$ and $\lim_{z\to\infty} H(z,c) = \infty$, we conclude that if $c^{\frac{1-\mu}{2}} \tilde{G}^{-(\alpha+\mu)} < 1$ then $H(z,c)$ intersects the horizontal axis $z = 0$ once and only once at some $\tilde{\tilde{z}}(c) > 0$.

The case with $c^{\frac{1-\mu}{2}}\tilde{G}^{-(\alpha+\mu)} = 1$ is similar to the case with $c^{\frac{1-\mu}{2}}\tilde{G}^{-(\alpha+\mu)} < 1$ with the only difference that $H(z,c)$ is an increasing function for all $z \neq \bar{\bar{z}}_0(c)$ and $H\left(\bar{\bar{z}}_0(c), c\right) = 0$, which means that $H(z,c)$ intersects the horizontal axis $z = 0$ only once at $\bar{\bar{z}}_0(c)$.

Now consider the case with $c^{\frac{1-\mu}{2}}\tilde{G}^{-(\alpha+\mu)} > 1$. We can write

$$H(z,c) = -c^{\frac{\mu-\alpha}{\alpha+\mu}} z^{-\frac{2(1-\alpha)}{2\alpha+\mu-1}} \tilde{H}\left(z^{-1}, c\right),$$

where

$$\tilde{H}(z,c) \equiv \tilde{G} z^{-\frac{2(1-\alpha)}{2\alpha+\mu-1}} - \frac{1-\alpha}{\alpha+\mu} c^{\frac{\alpha}{\alpha+\mu}} z^{-\frac{\mu+1}{2\alpha+\mu-1}} + \frac{1-\alpha}{\alpha+\mu} c^{-\frac{\mu}{\alpha+\mu}} z - c^{-\frac{\mu-\alpha}{\alpha+\mu}} \tilde{G}^{-1}.$$

Obviously, functions $H(z,c)$ and $\tilde{H}(z,c)$ have the same number of zeros for $z > 0$. Observe that function $\tilde{H}(z,c)$ is similar to function $H(z,c)$ with the difference that $\tilde{G}$ is swapped with $\tilde{G}^{-1}$, and c is swapped with c^{-1}. Applying the same analysis to function $\tilde{H}(z,c)$ as to function $H(z,c)$, we get that $\tilde{H}(z,c)$ increases in z if $c^{\frac{1-\mu}{2}}\tilde{G}^{-(\alpha+\mu)} > 1$. Then, given that $\lim_{z\to 0}\tilde{H}(z,c) = -\infty$ and $\lim_{z\to\infty}\tilde{H}(z,c) = \infty$, we conclude that $\tilde{H}(z,c)$ intersects the horizontal axis $z = 0$ once and only once for some $\tilde{\tilde{z}}(c) > 0$. The corresponding unique solution to equation $H(z,c) = 0$ is $[\tilde{\tilde{z}}(c)]^{-1}$.

At this point, we have established that for any $c > 0$ there is a unique $\tilde{\tilde{z}}(c) > 0$ such that $H(\tilde{\tilde{z}}(c), c) = 0$. This, of course, means that our original system $\tilde{V}(z, \phi_2, c) = 0$ and $\tilde{V}_1'(z, \phi_2, c) = 0$ has a unique solution $\left(\tilde{\tilde{z}}(c), \tilde{\tilde{\phi}}_2(c)\right)$ with $\tilde{\tilde{z}}(c) > 0$, where we use equation $\tilde{V}_1'(z, \phi_2, c) = 0$ to find $\tilde{\tilde{\phi}}_2(c)$ corresponding to $\tilde{\tilde{z}}(c)$, which gives

$$\tilde{\tilde{\phi}}_2(c) = \left(\frac{1-\alpha}{2\alpha+\mu-1}\left[\tilde{\tilde{z}}(c)\right]^{-\frac{\alpha+\mu}{2\alpha+\mu-1}} + \frac{\alpha+\mu}{2\alpha+\mu-1} c^{\frac{\mu}{\alpha+\mu}} \tilde{G}_2 \left[\tilde{\tilde{z}}(c)\right]^{\frac{1-\alpha}{2\alpha+\mu-1}}\right)^{-1}.$$

Obviously, $\tilde{\tilde{\phi}}_2(c) > 0$, while the upper bound $\tilde{\tilde{\phi}}_2(c) \leq \tilde{\phi}_2(c)$ simply follows from the fact that $\tilde{V}_1'\left(\tilde{\tilde{z}}(c), \tilde{\tilde{\phi}}_2(c), c\right) = 0$, and we know that $\tilde{V}_1'(z, \phi_2, c) < 0$ for all $z > 0$ and $\phi_2 > \tilde{\phi}_2(c)$.[2]

STEP 2. We are now going to prove the statement of Proposition 2 about the existence of a unique $\tilde{\tilde{\phi}}_2(c) > 0$ that traces the uniqueness boundary.

Consider $\tilde{V}_1'(z, \phi_2, c)$ for any $c > 0$ and $\phi_2 < \tilde{\phi}_2(c)$. We know that $\tilde{V}_1'(\bar{z}_0, \phi_2, c) > 0$ for $\phi_2 < \tilde{\phi}_2(c)$, where $\bar{z}_0(c)$ was defined in (13) and is the global maximum of $\tilde{V}_1'(z, \phi_2, c)$. This implies that $\tilde{V}_1'(z, \phi_2, c)$ intersects the horizontal axis $z = 0$ at exactly two points: one

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[2] As we have argued above, $\tilde{V}(z, \phi_2, c)$ is a decreasing function of z if $\phi_2 > \tilde{\phi}_2(c)$.

lower than $\bar{z}_0(c)$ and one larger than $\bar{z}_0(c)$. This fact allows us to define functions

$$z_1^*(\phi_2;c) \equiv \left\{z \middle| z \le \bar{z}_0(c) \text{ and } \tilde{V}_1'(z,\phi_2,c)=0\right\},$$
$$z_2^*(\phi_2;c) \equiv \left\{z \middle| z \ge \bar{z}_0(c) \text{ and } \tilde{V}_1'(z,\phi_2,c)=0\right\},$$

both with domain $\phi_2 < \tilde{\phi}_2(c)$ and parameterized by $c > 0$. Definitions of $z_1^*(\phi_2;c)$ and $z_2^*(\phi_2;c)$ imply that $z_1^*(\phi_2;c) < \bar{z}_0(c) < z_2^*(\phi_2;c)$. Moreover, we have that $\tilde{V}_1'(z,\phi_2,c) < 0$ for $z \in (0, z_1^*(\phi_2;c)) \cup (z_2^*(\phi_2;c), \infty)$ and $\tilde{V}_1'(z,\phi_2,c) > 0$ for $z \in (z_1^*(\phi_2;c), z_2^*(\phi_2;c))$. Therefore, $z_1^*(\phi_2;c)$ is a local minimum of $\tilde{V}(z,\phi_2,c)$ and $z_2^*(\phi_2;c)$ is a local maximum of $\tilde{V}(z,\phi_2,c)$, and $\tilde{V}(z_1^*(\phi_2;c),\phi_2,c) < \tilde{V}(z_2^*(\phi_2;c),\phi_2,c)$.

We have shown in Step 1 that there exists a unique solution $\left(\tilde{\tilde{z}}(c), \tilde{\tilde{\phi}}_2(c)\right)$ to the system of equations $\tilde{V}(z,\phi_2,c)=0$ and $\tilde{V}_1'(z,\phi_2,c)=0$. Moreover, the argument in Step 1 implies that $H(z,c) < 0$ if and only if $z < \tilde{\tilde{z}}(c)$. Simple algebra reveals that

$$H(\bar{z}_0(c),c) = \frac{\mu+1}{\alpha+\mu} c^{\frac{\mu-\alpha}{\alpha+\mu}} \tilde{G} \left(\left(c^{\frac{1-\mu}{2}} \tilde{G}^{-(\alpha+\mu)}\right)^{\frac{2}{\mu+1}} - 1\right),$$

and, thus, $H(\bar{z}_0(c),c) < 0$ if and only if $c^{\frac{1-\mu}{2}} \tilde{G}^{-(\alpha+\mu)} < 1$. Therefore, $\tilde{\tilde{z}}(c) > \bar{z}_0(c)$ if and only if $c^{\frac{1-\mu}{2}} \tilde{G}^{-(\alpha+\mu)} < 1$. This, in turn, implies that $\tilde{\tilde{z}}(c) = z_1^*\left(\tilde{\tilde{\phi}}_2(c);c\right)$ if $c^{\frac{1-\mu}{2}} \tilde{G}^{-(\alpha+\mu)} > 1$ and $\tilde{\tilde{z}}(c) = z_2^*\left(\tilde{\tilde{\phi}}_2(c);c\right)$ if $c^{\frac{1-\mu}{2}} \tilde{G}^{-(\alpha+\mu)} < 1$, while if $c^{\frac{1-\mu}{2}} \tilde{G}^{-(\alpha+\mu)} = 1$ then $\tilde{\tilde{z}}(c) = \bar{z}_0(c)$ and $\tilde{\tilde{\phi}}_2(c) = \tilde{\phi}_2(c)$.

Next, using the fact that $\tilde{V}_1'\left(z_i^*(\phi_2;c),\phi_2,c\right)=0$, we find that

$$\frac{d\tilde{V}\left(z_i^*(\phi_2;c),\phi_2,c\right)}{d\phi_2} = \left[z_i^*(\phi_2;c)\right]^{\frac{\alpha+\mu}{2\alpha+\mu-1}} \left(\left[z_i^*(\phi_2;c)\right]^{-\frac{\mu+1}{2\alpha+\mu-1}} - \left[\bar{z}_0(c)\right]^{-\frac{\mu+1}{2\alpha+\mu-1}}\right),$$

and, thus, $d\tilde{V}\left(z_i^*(\phi_2;c),\phi_2,c\right)/d\phi_2 > 0$ if and only if $z_i^*(\phi_2;c) < \bar{z}_0(c)$. This implies that $\tilde{V}(z_1^*(\phi_2;c),\phi_2,c)$ is increasing in ϕ_2, and $\tilde{V}(z_2^*(\phi_2;c),\phi_2,c)$ is decreasing in ϕ_2.

We are now ready to bring all facts together to characterize multiplicity of solutions of equation $\tilde{V}(z,\phi_2,c)=0$. Fix any $c>0$ such that $c^{\frac{1-\mu}{2}} \tilde{G}^{-(\alpha+\mu)} > 1$ and consider function $\tilde{V}(z,\phi_2,c)$ as we change ϕ_2. For $\phi_2 = \tilde{\tilde{\phi}}_2(c)$, the horizontal axis $z=0$ is tangent to the local minimum of $\tilde{V}(z,\phi_2,c)$ at point $\tilde{\tilde{z}}(c) = z_1^*\left(\tilde{\tilde{\phi}}_2(c);c\right)$. Thus, $\tilde{V}\left(z_1^*\left(\tilde{\tilde{\phi}}_2(c);c\right), \tilde{\tilde{\phi}}_2(c), c\right) = 0$ and for all points $z \in \left(0, z_2^*\left(\tilde{\tilde{\phi}}_2(c);c\right)\right)$ different from $z_1^*\left(\tilde{\tilde{\phi}}_2(c);c\right)$ we have $\tilde{V}\left(z,\tilde{\tilde{\phi}}_2(c),c\right) > 0$. For $z > z_2^*\left(\tilde{\tilde{\phi}}_2(c);c\right)$, function $\tilde{V}\left(z,\tilde{\tilde{\phi}}_2(c),c\right)$ monotonically decreases from a positive value to $-\infty$ as $z \to \infty$. This implies that function $\tilde{V}\left(z,\tilde{\tilde{\phi}}_2(c),c\right)$ crosses the horizontal axis $z=0$ only once for some $\tilde{z} > z_2^*\left(\tilde{\tilde{\phi}}_2(c);c\right)$.

Thus, for $\phi_2 = \tilde{\tilde{\phi}}_2(c)$ there are two solutions to equation $\tilde{V}(z, \phi_2, c) = 0$: $z_1^*\left(\tilde{\tilde{\phi}}_2(c); c\right)$ and $\tilde{z}$.

Next, as we have argued above, $\tilde{V}(z_1^*(\phi_2; c), \phi_2, c)$ is increasing in ϕ_2. Therefore, for $\phi_2 > \tilde{\tilde{\phi}}_2(c)$ we have $\tilde{V}(z_1^*(\phi_2; c), \phi_2, c) > 0$, which implies that $\tilde{V}(z, \phi_2, c) > 0$ for all $z \in (0, z_2^*(\phi_2; c))$. And for $z > z_2^*(\phi_2; c)$, again, function $\tilde{V}(z, \phi_2, c)$ intersects the horizontal axis $z = 0$ once and only once. Thus, for $\phi_2 > \tilde{\tilde{\phi}}_2(c)$ there is a unique solution to equation $\tilde{V}(z, \phi_2, c) = 0$.

Finally, for $\phi_2 < \tilde{\tilde{\phi}}_2(c)$ we have $\tilde{V}(z_1^*(\phi_2; c), \phi_2, c) < 0$. At the same time, we necessarily have $\tilde{V}(z_2^*(\phi_2; c), \phi_2, c) > 0$, because $\tilde{V}(z_2^*(\phi_2; c), \phi_2, c)$ is a decreasing function of ϕ_2 and $\tilde{V}\left(z_2^*\left(\tilde{\tilde{\phi}}_2(c); c\right), \tilde{\tilde{\phi}}_2(c), c\right) > 0$. Then, the facts that $\tilde{V}(0, \phi_2, c) > 0$ and that for any $\phi_2 < \tilde{\tilde{\phi}}_2(c)$ we have $\tilde{V}(z_1^*(\phi_2; c), \phi_2, c) < 0$ and $\tilde{V}(z_2^*(\phi_2; c), \phi_2, c) > 0$ imply that for $z \in (0, z_2^*(\phi_2; c))$ function $\tilde{V}(z, \phi_2, c)$ intersects the horizontal axis $z = 0$ exactly two times. In addition to that, as in the cases with $\phi_2 > \tilde{\tilde{\phi}}_2(c)$ and $\phi_2 = \tilde{\tilde{\phi}}_2(c)$, function $\tilde{V}(z, \phi_2, c)$ intersects the horizontal axis $z = 0$ one more time for some $\tilde{z} > z_2^*(\phi_2; c)$. Thus, for $\phi_2 < \tilde{\tilde{\phi}}_2(c)$ equation $\tilde{V}(z, \phi_2, c) = 0$ has three solutions.

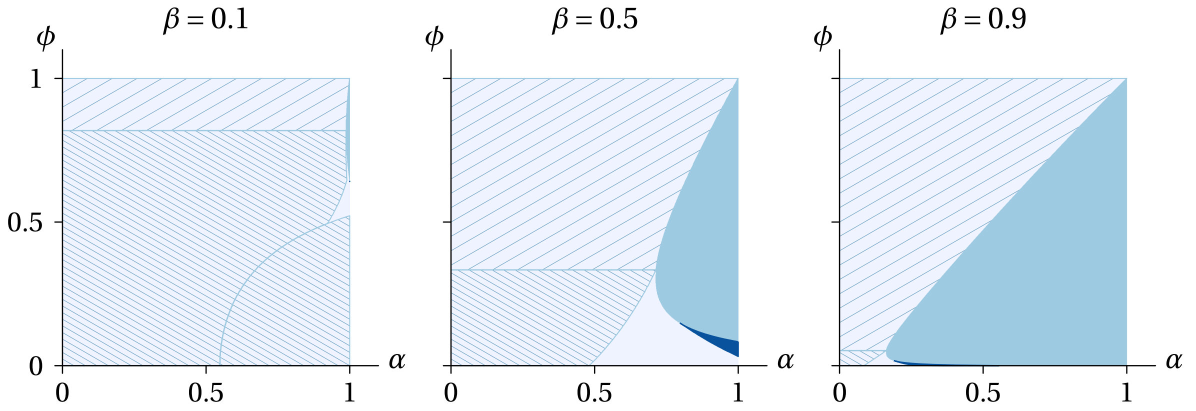
Analysis of multiplicity of solutions of $\tilde{V}(z, \phi_2, c)$ for $c^{\frac{1-\mu}{2}} \tilde{G}^{-(\alpha+\mu)} < 1$ is similar to the above analysis with $c^{\frac{1-\mu}{2}} \tilde{G}^{-(\alpha+\mu)} > 1$. The difference is that for $c^{\frac{1-\mu}{2}} \tilde{G}^{-(\alpha+\mu)} < 1$ the horizontal axis $z = 0$ is tangent to the local maximum of $\tilde{V}\left(z, \tilde{\tilde{\phi}}_2(c), c\right)$ at point $\tilde{\tilde{z}}(c) = z_2^*\left(\tilde{\tilde{\phi}}_2(c); c\right)$.

The case with $c^{\frac{1-\mu}{2}} \tilde{G}^{-(\alpha+\mu)} = 1$ is special. In this case $\tilde{\tilde{\phi}}_2(c) = \tilde{\phi}_2(c)$. To see this, observe that $\tilde{V}(\bar{z}_0(c), \phi_2, c) = 0$ for any ϕ_2. We know that for any $\phi_2 \geq \tilde{\phi}_2(c)$ equation $\tilde{V}(z, \phi_2, c) = 0$ has a unique solution. Thus, for all $\phi_2 \geq \tilde{\phi}_2(c)$ the unique solution to $\tilde{V}(z, \phi_2, c) = 0$ is $\bar{z}_0(c)$. For $\phi_2 < \tilde{\phi}_2(c)$, we have that $z_1^*(\phi_2; c) < \bar{z}_0(c) < z_2^*(\phi_2; c)$ and that $\tilde{V}_1'(z, \phi_2, c) < 0$ for $z \in (0, z_1^*(\phi_2; c)) \cup (z_2^*(\phi_2; c), \infty)$ and $\tilde{V}_1'(z, \phi_2, c) > 0$ for $z \in (z_1^*(\phi_2; c), z_2^*(\phi_2; c))$. Therefore, $\tilde{V}(z_1^*(\phi_2; c), \phi_2, c) < 0 < \tilde{V}(z_2^*(\phi_2; c), \phi_2, c)$. Then, given that $\tilde{V}(0, \phi_2, c) > 0$ and $\lim_{z \to \infty} \tilde{V}(z, \phi_2, c) = -\infty$, we conclude that $\tilde{V}(z, \phi_2, c)$ intersects the horizontal axis $z = 0$ once for $z < z_1^*(\phi_2; c)$ and once for $z > z_2^*(\phi_2; c)$. Thus, overall, for $\phi_2 < \tilde{\phi}_2(c)$, equation $\tilde{V}(z, \phi_2, c) = 0$ has three solutions (one of which is $\bar{z}_0(c)$).

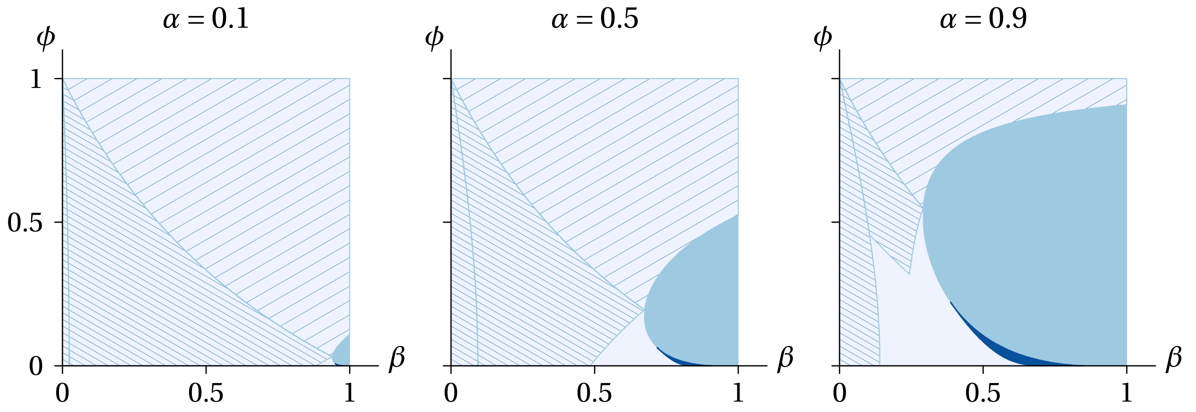
This concludes the proof of Proposition 2.

4 Role of Asymmetries in the Case with $0 < \alpha < 1$

In this section we focus on the role of asymmetries for the number of equilibria in the case with $0 < \alpha < 1$ and no input-output loops ($\zeta = 1$). The cases (ii.a)-(ii.b) of Proposition 3



(a) Varying $\phi \equiv \phi_1 = \phi_2$ and α.



(b) Varying $\phi \equiv \phi_1 = \phi_2$ and β.

▨ Unique equilibrium: Conditon (ii.a)

▨ Unique equilibrium: Conditon (ii.b)

☐ Unique equilibrium:
Not covered by conditions (ii.a)-(ii.b)

■ Three equilibria

■ Five equilibria

Figure 7: Uniqueness/multiplicity areas, $\varepsilon = 5$. Interaction of trade costs and parameters α and β in the symmetric case with $\phi_1 = \phi_2 \equiv \phi$, $\gamma = 0.5$, $G = 1$. Points on the boundaries between sets of one and three equilibria have a unique equilibrium. Points on the boundaries between sets of one and five equilibria have three equilibria. Points on the boundaries between sets of three and five equilibria have three equilibria.

from the main text as well as areas with one, three, and five equilibria are illustrated in Figures 7-11 for various model parameter values. In all of these figures, the vertical axis measures ϕ_1, while the horizontal axis measures one of the other parameters: α, β, ϕ_2, γ or G. For all of these figures we set $\varepsilon = 5$.[3] Also, differently from the main text, we do not vary μ independently from other parameters: given values of α, β, and ε, we set

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[3] Figures for other values of ε are very similar.

μ according to its definition for the case with no input-output loops, $\mu = \beta \cdot (\alpha/\varepsilon + 1)$. Figure 7 corresponds to the symmetric case with $\phi_1 = \phi_2 \equiv \phi$, $\gamma = 0.5$, and $G = 1$, while Figures 8-11 illustrate the consequences of asymmetries in trade costs and parameters γ and G.

As discussed in Section 3 of the main text, the number of equilibria for a particular set of parameters is unambiguously determined by the number of critical points of $V(x)$ — given by condition $V'(x) = 0$ — and the sign pattern of $V(x)$ evaluated at these critical points, where $V(x)$ is the equilibrium function defined in the main text. Following this approach, we find that parameters strictly inside the sets of one, three, and five equilibria in Figures 7-11 yield solutions to $V(x) = 0$ that satisfy $V'(x) \neq 0$, while parameters on the boundaries between these sets yield solutions to $V(x) = 0$ that satisfy $V'(x) = 0$. Thus, the boundary between sets with five equilibria and sets with three or one equilibria generically has four equilibria, while the boundary between sets with three equilibria and sets with one equilibrium generically has two equilibria.[4]

Let us start with Figure 7 showing outcomes in the symmetric case. This figure summarizes some of the results that we discuss in the main text. Namely, uniqueness of equilibria depends, among other things, on the interaction of trade costs and parameters α and β, so that the sufficient condition for uniqueness $\alpha \leq 0$ used — following Allen and Arkolakis (2014) — in the recent economic geography literature is particularly strong and somewhat conceals the complexity of the outcomes in the case with $0 < \alpha < 1$. Another message of Figure 7 — discussed in Sections 4 and 5 of the main text — is that lower values of α and/or β tend to result in a bigger set of trade costs for which the economy has a unique equilibrium. A fresh insight coming out of Figure 7 is that each of the sufficient conditions (ii.a)-(ii.b) of Proposition 3 from the main text is relevant for some parameter values, and there are areas of uniqueness not covered by any of these conditions. In other words, the sufficient conditions in Proposition Proposition 3 from the main are not necessary.

Figure 7 seems to suggest that lower values of α and/or β are more likely to result in uniqueness. We see from panel (a) of Figure 7 that, for each level of trade costs and each

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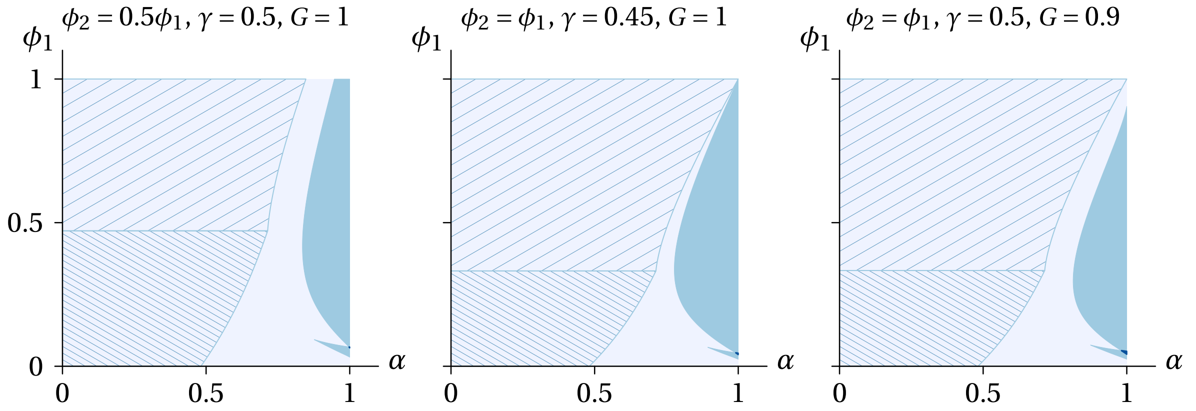
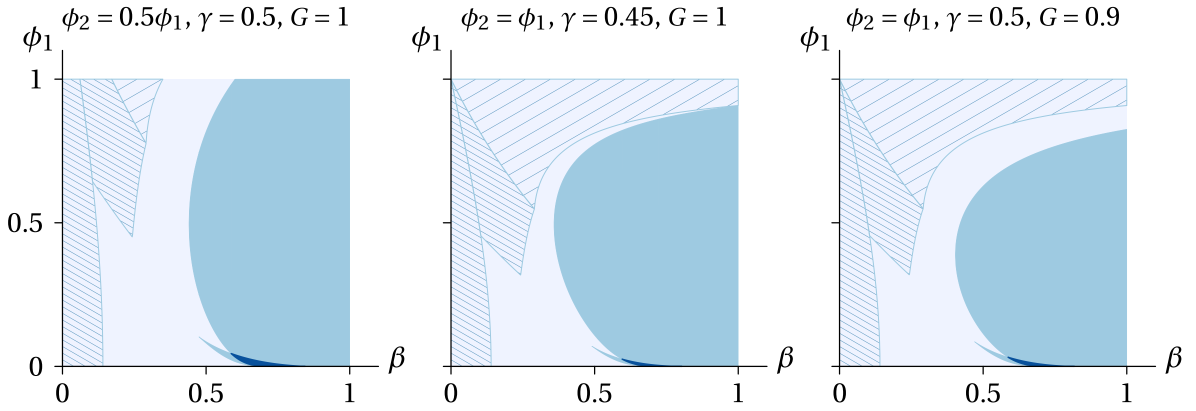
[4] Exceptions are the cases when several critical points of $V(x)$ simultaneously satisfy $V(x) = 0$. In the general asymmetric case, this can happen only at a finite number of points. In such cases, the boundary between sets with five and sets with three or one equilibria can have three equilibria, while the boundary between sets with three and one equilibria can have one equilibrium. Since Figure 7 corresponds to the symmetric case with $\phi_1 = \phi_2 \equiv \phi$, $\gamma = 0.5$, and $G = 1$, the boundaries between sets of one, three, and five equilibria feature an odd number of equilibria, while in Figures 8-11 the boundaries generically feature an even number of equilibria. An odd number of equilibria on the boundaries in the symmetric case follows from the fact that in the symmetric case, for each x^* satisfying $V(x) = 0$ and $V'(x) = 0$, we have that $[x^*]^{-1}$ also satisfies the same conditions. Thus, in the symmetric case, conditions $V(x) = 0$ and $V'(x) = 0$ are always satisfied for a pair of critical points.



(a) Varying ϕ_1 and α with $\beta = 0.5$.

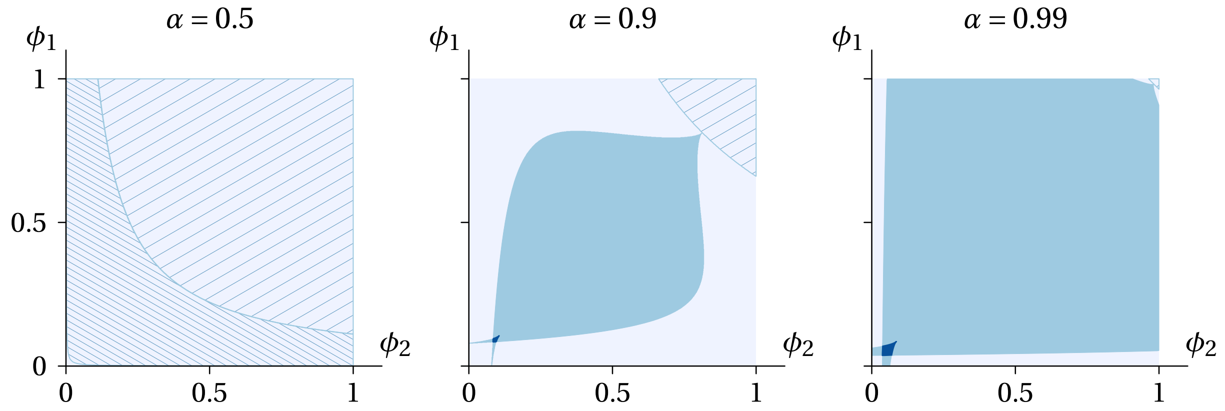


(b) Varying ϕ_1 and β with $\alpha = 0.9$.

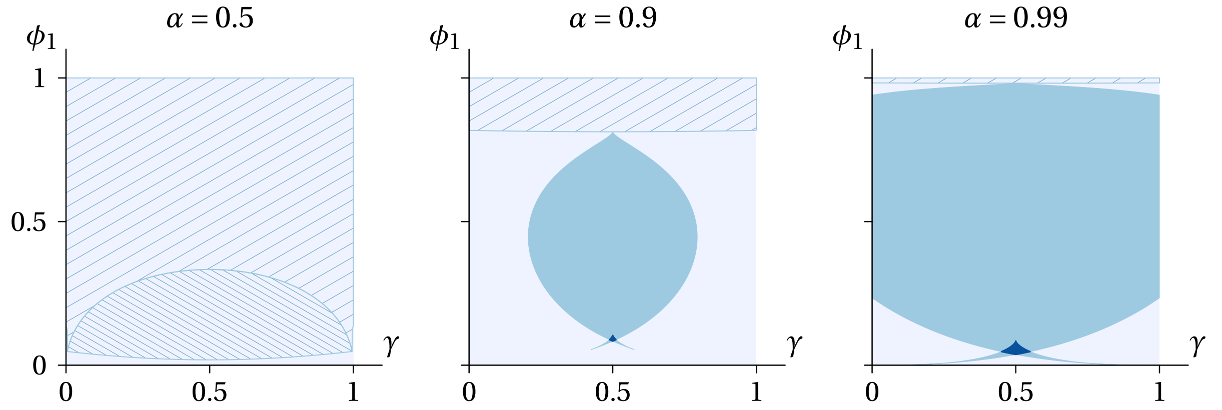
Figure 8: Uniqueness/multiplicity areas, $\varepsilon = 5$. Interaction of trade costs and parameters α and β in the case of asymmetries in only one characteristic of regions. Legend is the same as in Figure 7. Points on the boundaries between sets of one and three equilibria generically have two equilibria. Points on the boundaries between sets of one and five equilibria generically have four equilibria. Points on the boundaries between sets of three and five equilibria generically have four equilibria.

value of parameter β, either the equilibrium is unique for all $\alpha \in (0, 1)$, or there is a threshold value $\hat{\alpha}$ such that the equilibrium is unique if and only if $\alpha < \hat{\alpha}$. That is, by increasing α and keeping everything else fixed, we can only go from the region of uniqueness to the region of multiplicity, and never in reverse. Panel (b) of Figure 7 shows a similar behavior in terms of β. Such monotonic behavior, however, is, if anything, a consequence of the symmetry between regions and is generally not true. This is demonstrated in Figure 8.

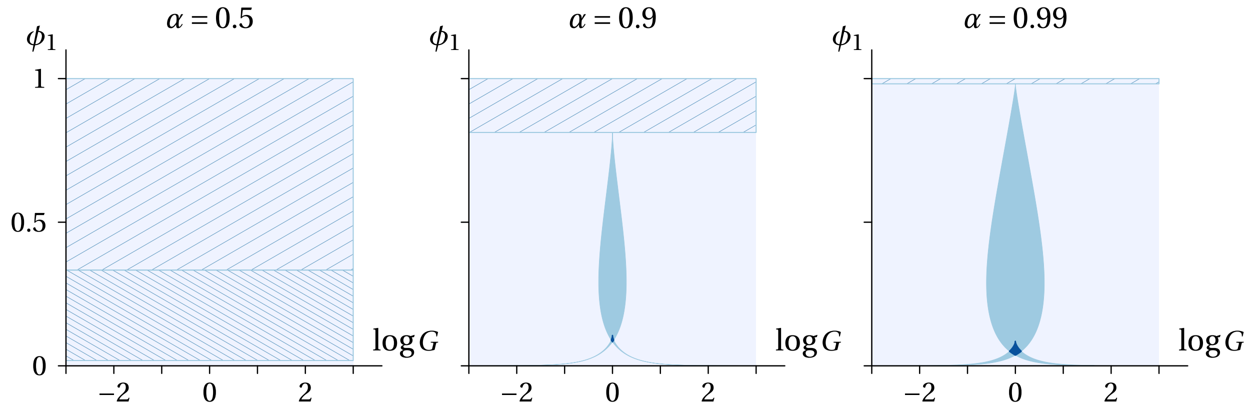
Panel (a) of Figure 8 shows the number of equilibria in (α, ϕ_1) coordinates for $\beta = 0.5$ and for three types of asymmetries: the left picture of panel (a) corresponds to asymmetries in trade costs between regions given by $\phi_2/\phi_1 = 0.5$ (implying that it is costlier to export from region 1 to region 2 than in the opposite direction); the central picture of



(a) Varying ϕ_1 and ϕ_2 with $\gamma = 0.5$ and $G = 1$.



(b) Varying ϕ_1 and γ with $\phi_2 = \phi_1$ and $G = 1$.



(c) Varying ϕ_1 and G with $\phi_2 = \phi_1$ and $\gamma = 0.5$.

Figure 9: Uniqueness/multiplicity areas, $\varepsilon = 5$ and $\beta = 0.5$. Interaction of trade costs and characteristics of regions: allowing asymmetry in only one characteristic at a time. Legend is the same as in Figure 7. Points on the boundaries between sets of one and three equilibria generically have two equilibria. Points on the boundaries between sets of one and five equilibria generically have four equilibria. Points on the boundaries between sets of three and five equilibria generically have four equilibria.

panel (a) corresponds to asymmetries in agricultural labor endowments given by $\gamma = 0.45$ (implying more agricultural labor in region 2); and the right picture of panel (a) corresponds to asymmetries in productivities and/or amenity endowments of regions given by $G = 0.9$ (implying higher productivity/amenities in region 2). Panel (b) of Figure 8 is similar to panel (a) with the difference that panel (b) shows the number of equilibria in (β, ϕ_1) coordinates for $\alpha = 0.5$. As we can see from Figure 8, increasing α or β, and keeping everything else fixed, generally can result in intermitting regions of uniqueness and multiplicity.

Comparing pictures in panel (a) of Figure 8 with the picture corresponding to $\beta = 0.5$ in panel (a) of Figure 7, we see that introduction of asymmetries in just one characteristic of regions tends to increase uniqueness areas in the (α, ϕ_1) space. A similar observation can be made with regard to the (β, ϕ_1) space — compare pictures in panel (b) of Figure 8 with the picture corresponding to $\alpha = 0.9$ in panel (b) of Figure 7.

Figure 9 further reinforces the two points made above: lower values of α as well as asymmetries in one characteristic of regions tend to result in a larger set of uniqueness outcomes. In this figure, we set $\beta = 0.5$ and $\alpha = 0.5$, 0.9 and 0.99, and show the uniqueness/multiplicity areas as we vary the trade freeness parameter ϕ_1 and one of the three other parameters: ϕ_2, γ or G.

As we can see from Figure 9, all equilibria are unique in all pictures corresponding to $\alpha = 0.5$, while the set of unique equilibria is the smallest in pictures corresponding to $\alpha = 0.99$. However, the dependence of the size of the uniqueness area on parameter α is not necessarily monotonic. We also see from panel (a) of Figure 9 that making trade costs asymmetric (while keeping the mean trade freeness $(\phi_1 \phi_2)^{1/2}$ fixed) tends to result in uniqueness, but this behavior is not always monotonic. Observe the splitting "tail" of the region of uniqueness for low values of ϕ_1 and ϕ_2. Finally, panels (b) and (c) of Figure 9 show that making the regions asymmetric by taking parameters γ or G to their extreme values tends to result in uniqueness for a given value of ϕ_1. However, again, this behavior is not always monotonic.

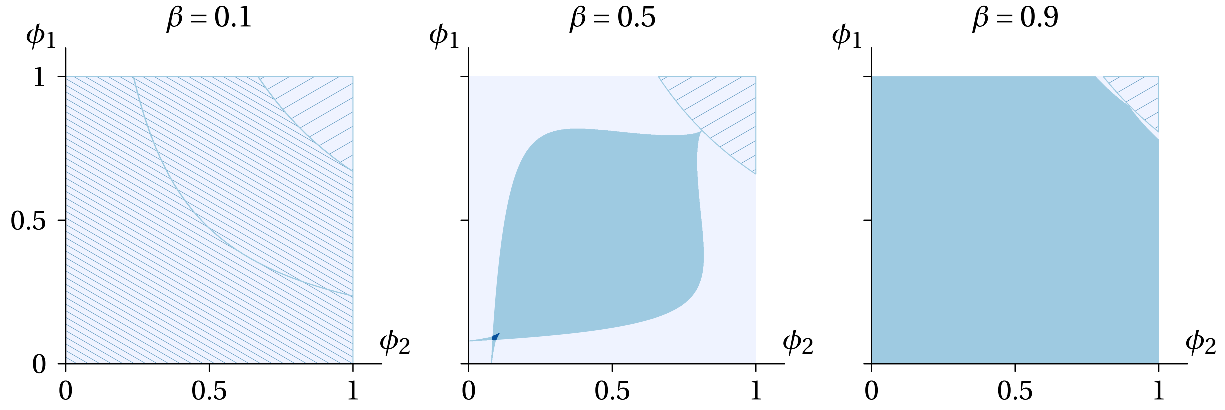
Figure 10 is similar to Figure 9 and shows the uniqueness/multiplicity areas for $\alpha = 0.9$ and $\beta = 0.1$, 0.5 and 0.9. The message is the same as that of Figure 9: lower values of β as well as asymmetries in one characteristic of regions tend to result in a larger set of uniqueness outcomes.

Figure 11 shows the consequences of introducing asymmetries in two or all three region characteristics. In this figure, we fix $\alpha = 0.9$ and $\beta = 0.5$. Consider panel (a) of this figure, which shows uniqueness/multiplicity areas in the (ϕ_1, ϕ_2) space for three sets of values of parameters γ and G: $\gamma = 0.2$ and $G = 1$ (asymmetries only in the agricultural labor endowments); $\gamma = 0.5$ and $G = 1.5$ (asymmetries only in productivities/amenities); and $\gamma = 0.2$ and $G = 1.5$ (asymmetries both in agricultural labor endowments and productivities/amenities). Observe how in the picture corresponding to $\gamma = 0.2$ and $G = 1$ the multiplicity region shifts to the bottom and to the right relative to the multiplicity region in the case with $\gamma = 0.5$ and $G = 1$ shown in the central picture in panel (a) of Figure 9. This outcome is intuitive. Having less agricultural labor in region 1 ($\gamma = 0.2 < 1$) combined with relatively high costs of shipping manufactured goods from region 1 to region 2 (low values of ϕ_2/ϕ_1) makes region 2 a more attractive place to concentrate manufacturing production, which reduces indeterminacy of outcomes and, thus, tends to result in uniqueness. At the same time, when costs of shipping goods from region 1 to region 2 are relatively low, region 1 can be a potentially attractive location for concentration of manufacturing production despite low agricultural labor endowment. This creates indeterminacy of outcomes and potentially leads to multiplicity.

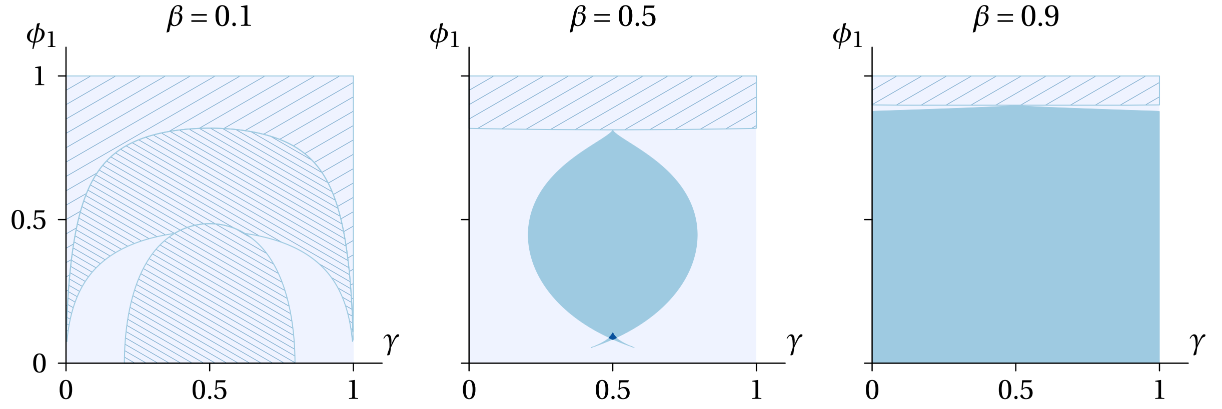
Next, consider the case with $\gamma = 0.5$ and $G = 1.5$ in panel (a) of Figure 11. In this case region 1 has larger productivity/amenities. Relative to the case with $\gamma = 0.5$ and $G = 1$, the multiplicity region shifts up and to the left in the (ϕ_1, ϕ_2) space, which is, again, intuitive — the explanation is similar to the one for the case with $\gamma = 0.2$ and $G = 1$.

More interesting is the case with $\gamma = 0.2$ and $G = 1.5$, where asymmetries in the agricultural labor endowments and productivities/amenities work against each other: region 1 is more attractive due to its productivity/amenities, while region 2 is more attractive due to its agricultural labor abundance. Introduction of asymmetries in trade costs on top of this reinforces attractiveness of one region and diminishes attractiveness of the other region. This creates indeterminacy of outcomes for relatively small asymmetries in trade costs, but reduces indeterminacy of outcomes for large asymmetries in trade costs. This explains why the multiplicity region expands to both sides of the line $\phi_1 = \phi_2$ relative to the case with $\gamma = 0.5$ and $G = 1$.

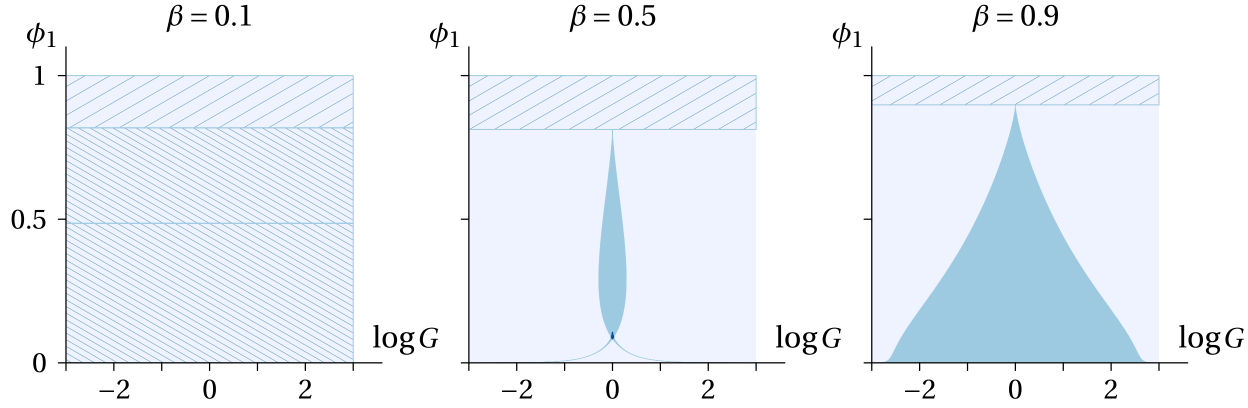
The pictures in panels (b) and (c) of Figure 11 show how the uniqueness/multiplicity areas shift in the (ϕ_1, γ) and $(\phi_1, \log G)$ spaces. The explanation is conceptually the same as for the panel (a). The upshot of this analysis is that making regions more asymmetric in several characteristics can make a uniqueness outcome more or less likely depending on whether asymmetries in characteristics favor one of the regions or make these regions similarly attractive for concentration of manufacturing production.



(a) Varying ϕ_1 and ϕ_2 with $\gamma = 0.5$ and $G = 1$.

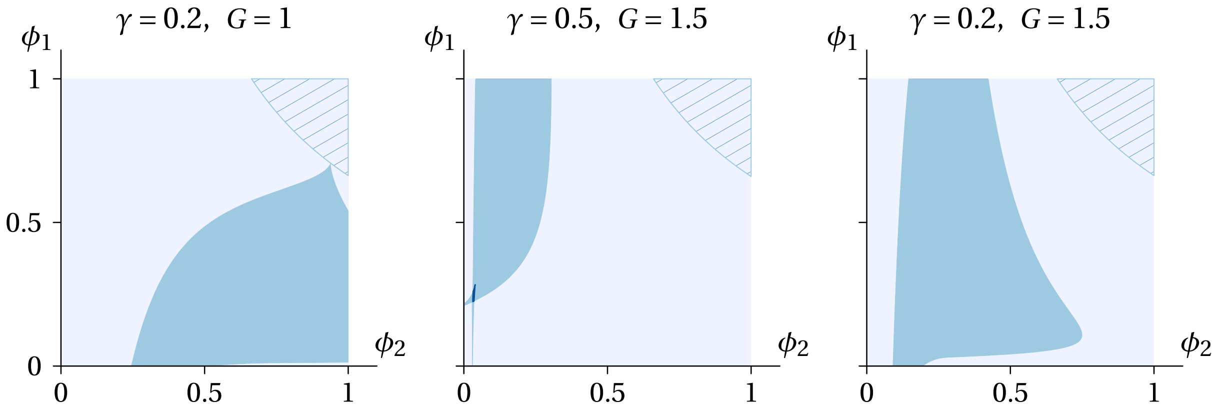


(b) Varying ϕ_1 and γ with $\phi_2 = \phi_1$ and $G = 1$.

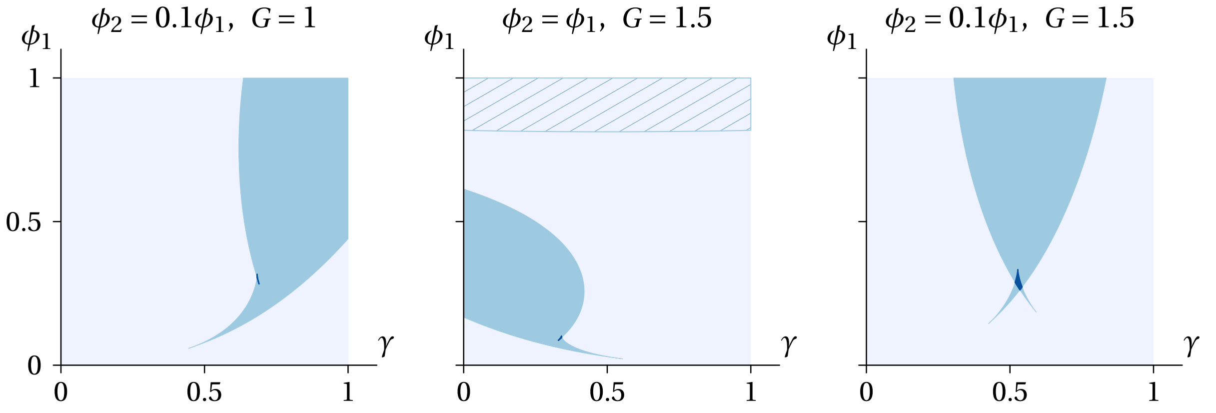


(c) Varying ϕ_1 and G with $\phi_2 = \phi_1$ and $\gamma = 0.5$.

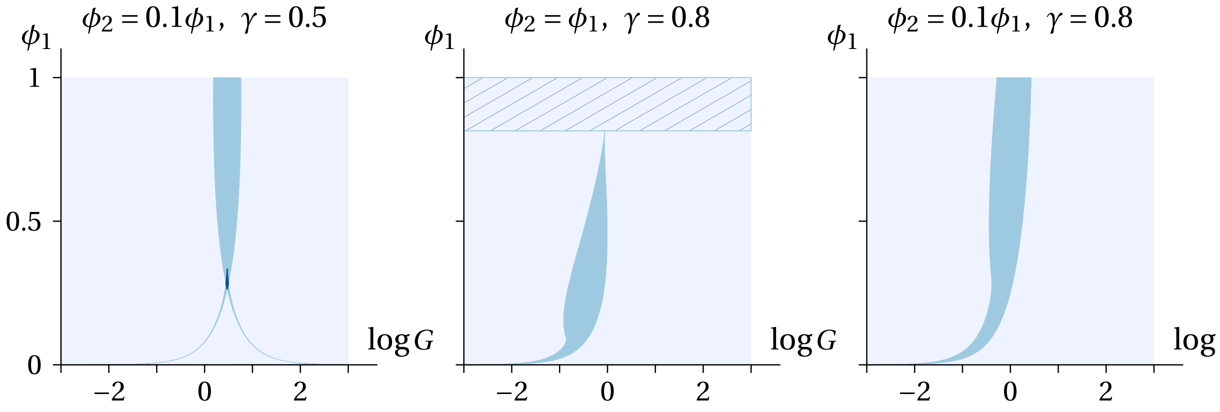
Figure 10: Uniqueness/multiplicity areas, $\varepsilon = 5$ and $\alpha = 0.9$. Interaction of trade costs and characteristics of regions: allowing asymmetry in only one characteristic at a time. Legend is the same as in Figure 7. Points on the boundaries between sets of one and three equilibria generically have two equilibria. Points on the boundaries between sets of one and five equilibria generically have four equilibria. Points on the boundaries between sets of three and five equilibria generically have four equilibria.



(a) Varying ϕ_1 and ϕ_2.



(b) Varying ϕ_1 and γ.



(c) Varying ϕ_1 and G.

Figure 11: Uniqueness/multiplicity areas, $\varepsilon = 5$, $\alpha = 0.9$, and $\beta = 0.5$. Interaction of trade costs and characteristics of regions: allowing asymmetries in two or all three characteristics. Legend is the same as in Figure 7. Points on the boundaries between sets of one and three equilibria generically have two equilibria. Points on the boundaries between sets of one and five equilibria generically have four equilibria. Points on the boundaries between sets of three and five equilibria generically have four equilibria.

References

ALLEN, T. and ARKOLAKIS, C. (2014). Trade and the Topography of the Spatial Economy. *The Quarterly Journal of Economics*, **129** (3), 1085–1140.

KEHOE, T. J. (1980). An Index Theorem for General Equilibrium Models with Production. *Econometrica*, **48** (5), 1211–32.